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Archimedes, the puck, and the turntable

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Abstract

A standard problem in mechanics asks us to look at a frictionless puck sliding over and across a turntable (a rotating platform with constant angular velocity) from the rotating reference frame S' of the turntable. Here we analyse in some detail the kinematics and dynamics of such a puck, where a type of spiral discovered by Archimedes plays a central role.

A puck is pushed with speed v_o from the edge of a circular turntable of radius R towards its centre, while the turntable rotates with constant angular velocity ω. From the inertial 'lab' frame S the puck is seen to move in a straight horizontal line until it emerges at the opposite end of the diameter, as the net force (weight plus normal) is zero. From the viewpoint of an observer in the turntable frame, S', things look quite different. This situation has been presented in intermediate mechanics textbooks, for example in Taylor [1]; one exercise is qualitative, while the other asks for the motion in S' to be expressed in polar coordinates. The instructor's manual [2] solves this problem and sketches a sample path.

We now examine the motion of the puck in the S' frame in more detail. First, we divide the puck's motion into two legs, the first from edge to centre and the second from centre to edge. Each half of the motion takes $t = R/v_o$.

The initial angle of the puck is $\varphi = 0$. The motion can be described in polar coordinates as shown below. These equations follow from those of relative motion:

for $t \leqslant R/v_o$**:**

$$r' = R - v_0 t, \tag{1a}$$

$$\varphi' = -\omega t; \tag{1b}$$

for $R/v_o < t < 2R/v_o$**:**

$$r' = v_0 t - R, \tag{2a}$$

$$\varphi' = \pi - \omega t. \tag{2b}$$

This is somewhat different from the answer provided in Taylor's solution manual, which has $\varphi' = -\omega t$ for all times, but allows for a negative radius for $t > R/v_o$.

Sample trajectories are shown in figure 1. For purposes of illustration, we let $\omega = \pi$ rad s^{-1} and $R = 1$ length unit. The paths were obtained using the Python code given in the appendix. Panel (a) shows the straight-line motion in frame S ($\omega = 0$). Panels (b)–(h) show the motion as seen in frame S' with decreasing linear velocities of the puck, resulting in increasingly complex trajectories. The path in panel (b) is very similar to that shown in [2].

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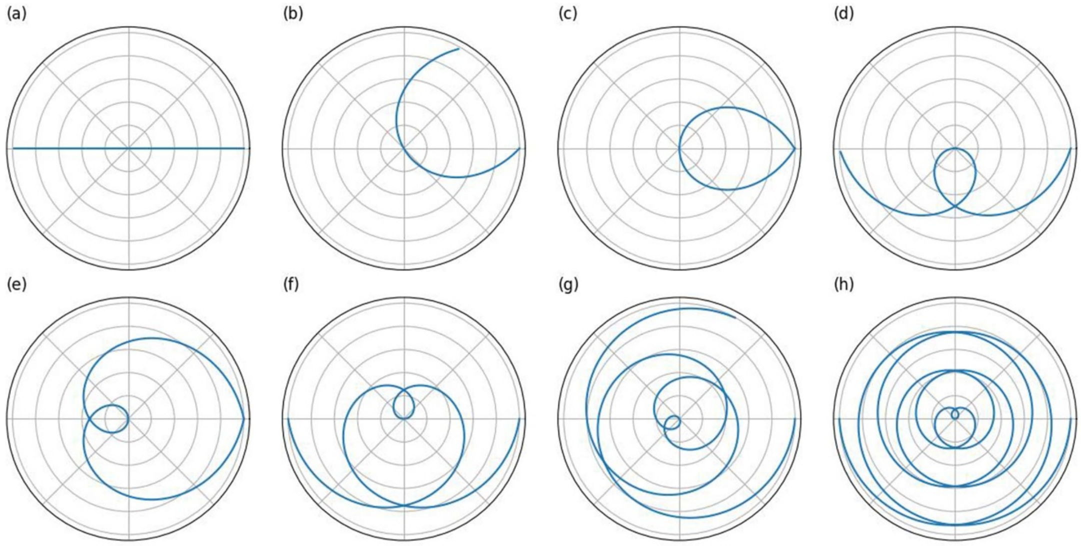

Figure 1. Motion of a puck over a counterclockwise-rotating turntable, with $R = 1$ length unit and $\omega = \pi$ rad s^{-1}. The puck's entry point is always at right centre. Panel (a) shows the straight-line motion in fixed frame S. The other panels show the trajectories in rotating frame S' with different values of puck speed v_o in length units s^{-1}: (b) 3.0; (c) 2.0; (d) 1.0; (e) 2/3; (f) 1/2; (g) 0.3; (h) 1/6, in length units s^{-1}. Note the mirror symmetry of the trajectories.

One can eliminate time from equations (1*a*), (1*b*) and (2*a*), (2*b*). The resulting trajectories are of the form $r = a + b\varphi$, with b positive during the inward leg and negative during the outward leg, as the angle becomes more and more negative. The trajectories are clearly spirals, and the sign of b determines the chirality ('handedness') of the respective spirals, a property that plays a role in astronomy [3], chemistry [4], and biology [5].

Out of the two dozen or so [6, 7] named spiral types, this form corresponds to the Archimedes' spiral, discovered by the famous Greek mathematician of ancient times [8]. Note that the trajectories exhibit mirror symmetry about angle $\varphi' = \omega R/v_o$, consistent with the change in chirality. The spiral nature of the trajectories is hard to infer from large v_o cases, such as panels (b) and (c) in figure 1 and more evident in low-speed cases (e)–(h).

While in frame S the speed is constant ($v_x = -v_o$ in Cartesian coordinates) and the acceleration is zero, one gets different results in the S' frame.

From [1] we have, in polar coordinates,

$$\boldsymbol{v} = \dot{r}\hat{r} + r\dot{\varphi}\hat{\varphi},$$
$$\boldsymbol{a} = \left(\ddot{r} - r\dot{\varphi}^2\right)\hat{r} + \left(r\ddot{\varphi} + 2\dot{r}\dot{\varphi}\right)\hat{\varphi}. \qquad (3)$$

There is no radial or tangential acceleration in this case; $\dot{\varphi} = -\omega$, and $\dot{r} = \mp v_0$. Hence, for the inward and outward legs, we have:

$$\boldsymbol{v} = \mp v_0\hat{r} - \omega r\hat{\varphi},$$
$$\boldsymbol{a} = \left(-r\omega^2\right)\hat{r} \pm 2v_0\omega\hat{\varphi}. \qquad (4)$$

The results presented are appropriate for several student audiences. For those who do not intend to study more formal physics, the plots of equations (1) and (2) shown in figure 1, which can be derived from relative motion, illustrate the complexity of rotating reference frames. This will be useful if the students are enrolled in a nontechnical earth or atmospheric sciences program, where our planet's rotating reference frame is often a desirable choice. See also the qualitative treatment in [9]. For students in a more advanced mechanics class, equations (3) and (4) will eventually be encountered, and the plots can be used to discuss the interplay between the centrifugal and Coriolis accelerations—the two surviving terms in equation (4). These two accelerations, and their respective forces, are fictitious as they do not have a physical agent.

Data availability statement

No new data were created or analysed in this study.

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Conflict of interest

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Appendix. Python code (single panel)

```
import numpy as np
import matplotlib.pyplot as plt
import math

#Variables: v = puck speed; o, r: turntable angular velocity and radius
#Variables: t = elapsed time; increment = time increment for loop
#Variables: thetas and rads = angular and radial coordinate

increment = 0.01
v = 0.5
r = 1
o = math.pi

rads = []
thetas = []

#Time loop : New data are appended to a list

t = 0
while t <= r/v: #First leg (towards centre)
    rads.append(r-v*t) #Equation (1a)
    thetas.append(-o*t) #Equation (1b)
    t += increment

while t < 2*(r/v): #Second leg (away from centre)
    rads.append(v * t - r) #Equation (2a)
    thetas.append(math.pi - o * t) #Equation (2b)
    t += increment

#The next two lines generate the polar plot

plt.polar(thetas, rads)

plt.show()
```

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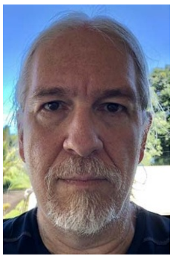

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Keith Lyons dropped out of high school after a near-fatal car crash damaged his memory. He spent the next fifteen years building computers, fixing cars, and working as a handyman. He moved to Hawaii and did tool rental and repair until another near-fatal car crash left him unable to work. He then returned to school, and is currently pursuing an astronomy degree at the University of Hawaii.

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