Evolving A* to Efficiently Solve the κ Shortest-Path Problem (Extended Version)

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Abstract

The problem of finding the shortest path in a graph $G(V, E)$ has been widely studied. However, in many applications it is necessary to compute an arbitrary number of them, κ. Even though the problem has raised a lot of interest from different research communities and many applications of it are known, it has not been addressed to the same extent as the single shortest path problem. The best algorithm known for efficiently solving this task has a time complexity of $O(|E| + |V| \log |V| + \kappa|V|)$ when computing paths in explicit form, and is based on best-first search. This paper introduces a new search algorithm with the same time complexity, which results from a natural evolution of A* thus, it preserves all its interesting properties, making it widely applicable to many different domains. Experiments in various testbeds show a significant improvement in performance over the state of the art, often by one or two orders of magnitude.

1 Introduction

Given a graph $G(V, E)$, the problem of finding the shortest path between two designated vertices s and t is a long-studied task, and A* (P. E. Hart et al., 1968) is a prominent algorithm used to solve it. A natural extension consists of computing the best κ paths[1] between the same vertices. David Eppstein (Eppstein, 1998) provides a thorough review in the history of the research on this task, noting that it dates back as far as 1957. Many variants have been considered, differing on various criteria, such as whether paths are required to be simple (or loopless) or whether the graphs considered are directed or undirected. This paper focuses on the problem of finding the κ, not necessarily simple, shortest paths between a start state, s, and a goal state, t, in directed graphs.

The problem has been already addressed with various heuristic search algorithms, usually with various derivative versions. mA* (Dechter et al., 2012; Flerova et al., 2016) is a straightforward application of A* which allows the expansion of nodes up to κ times. Doing so clearly allows the discovery of κ paths, and the idea can be easily applied to different domains. In contrast, K* (Aljazzar, 2009; Aljazzar & Leue, 2011) expands nodes only once. It is a heuristic variant of Eppstein's algorithm (EA) (Eppstein, 1998) which, in addition, can be built on-the-fly significantly improving its running time. K* essentially transforms EA to return paths as soon as practical. At the center of both EA and K* is the *path graph*, a structure which stores information from the search allowing the enumeration of paths through a one-to-one mapping between paths in the path graph and paths in the true graph. The algorithm swaps between search and enumerating paths from the path graph based on some swapping criterion, which can lead to the algorithm expanding nodes unnecessarily. The algorithm has been recently modified (Katz & Lee, 2023) with a variety of improvements, including a modification of the swapping criterion. Still, both EA and K* have an algorithmic complexity equal to $O(|E| + |V| \log |V| + \kappa)$ when outputting paths in *implicit* form, i.e., as a sequence of *sidetrack* edges. Usually, however, paths are required in *explicit* form, i.e., as a sequence of *vertices* and their algorithm complexity is then $O(|E| + |V| \log |V| + \kappa|V|)$.

In this paper, a novel search algorithm, BELA* (*Bidirectional Edge Labeling A**), is introduced. Some relevant definitions are introduced first and, among them, a novel use of sidetrack edges is proposed which splits paths into two components. At the core of our contribution is the notion of a *centroid* which we then use for the introduction of the brute-force variant of our algorithm, BELA_0. Its theoretical properties are examined and its algorithmic complexity studied. We then consider the heuristic version of the algorithm, BELA*. Afterwards, through empirical evaluation, we show BELA_0 and BELA* outperform both mA* and K* (as well as their brute-force variants), in a wide selection of problems often by one or two orders of magnitude in running time, and sometimes even more.

2 Definitions

Given a *directed* graph $G(V, E)$ characterized by its set of vertices $v \in V$ and edges $e_{ij} : v_i \rightarrow v_j, e_{ij} \in E$, let s and t denote the *start* and *goal* vertices respectively, between which an arbitrary number κ of shortest-paths has to be

[1]The letter k, commonly used for referring to the number of paths to find, is used throughout this paper as a generic index instead.

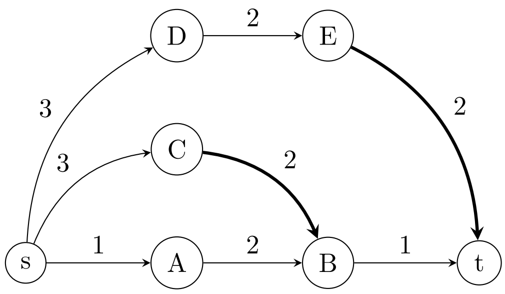


Figure 1: Examples of sidetrack edges —shown in thick lines

found. A path π is defined as a concatenation of vertices $\pi\langle n_0, n_1, n_2, \ldots, n_k\rangle$ such that $e_{n_{i-1},n_i} \in E, 0 < i \leq k$. If $s = n_0$ and $t = n_k$ then π is denoted as a *solution path*. The edges are weighted with non-negative integers, where $\omega(n_{i-1}, n_i)$ denotes the cost of traversing the edge e_{n_{i-1},n_i}. Thus, the cost of a path π is defined according to the *additive model* as $C(\pi) = \sum_{i=1}^{k} \omega(n_{i-1}, n_i)$. When the path π is clear from the context or it is irrelevant, the same cost can be denoted also as $g(n_i)$ if and only if the path starts at the start vertex, s. Analogously, $g_b(n_i)$ denotes the cost of the path from n_i to t computed as $g_b(n_i) = \sum_{j=i+1}^{k} \omega(n_{j-1}, n_j)$ if and only if $n_k = t$. A path π is said to be *optimal*, and is denoted as π^*, if and only if $C(\pi^*) \leq C(\pi')$ for every solution path π' between s and t, and is termed as *suboptimal* otherwise. Following the previous definitions, $g^*(n_i)$, and $g_b^*(n_i)$ denote the cost of an optimal path from the start vertex, s to n_i, and from n_i to t, respectively.

Heuristic functions are denoted as $h(\cdot)$. A heuristic function is said to be *admissible* if and only if $h(n) \leq h^*(n)$ for every node n, where $h^*(n)$ denotes the cost of an optimal solution from n to the goal t. Note that this definition refers to *nodes* instead of vertices, which are defined in turn, as the representation of a unique path from s to it, so that the same vertex can be represented with multiple nodes in a search algorithm. A heuristic function is said to be *consistent* if and only if $h(n) - h(n_i) \leq \omega(n, n_i)$, for every node n, where n_i is any descendant of it.

The set of all solution paths (either optimal or suboptimal) in G is denoted as G_π, and the set of all paths which are suboptimal is denoted as G'_π, $G'_\pi \subset G_\pi$.

Definition 1. *Given a directed $G(V, E)$ potentially infinite locally finite graph with natural edge weights, and two designated vertices, $s, t \in V$, the* single-source κ shortest-path problem *consists of finding a set of different, not necessarily simple paths $\Pi = \{\pi_0, \pi_1, \ldots, \pi_{\kappa-1}\}$ such that:*

- *If there exists a path π' such that $C(\pi') < C(\pi_i), 0 \leq i < \kappa$, then $\pi' \in \Pi$*
- *If $|G_\pi| \leq \kappa$, then $\Pi = G_\pi$*

Every solution path $\pi_i \in \Pi$ has a cost possibly different than the cost of other accepted solution paths. C_0^* represents the cost of all optimal solution paths $\pi_i \in G_\pi \backslash G'_\pi$; C_1^* is the cost of the cheapest suboptimal solution, $C_1^* > C_0^*$. Likewise, C_i^* is the cost of all suboptimal solution paths which are the i-th best, and C_φ^* represents the cost of the worst solutions in Π. In particular, $C(\pi_{\kappa-1}) = C_\varphi^*$.

Eppstein's Algorithm (EA) classified all edges in a graph in two different categories: *tree edges* and *sidetrack edges*, and K* slightly modified the definition of the second term. In the following, we adhere to the definitions used in K*:

Definition 2. *An edge e_{n_i,n_j} is a* tree edge *if and only if $g^*(n_j) = g^*(n_i) + \omega(n_i, n_j)$, and is said to be a* sidetrack *edge otherwise, i.e., $g^*(n_j) < g^*(n_i) + \omega(n_i, n_j)$.*

Clearly, the existence of at least one sidetrack edge is both a necessary and sufficient condition for a path to be suboptimal. One of our core contributions is that they also provide a means for distinguishing different components of any suboptimal solution path:

Definition 3. *Given a directed cyclic graph $G(V, E)$ with natural edge weights, and two designated vertices, $s, t \in V$, any suboptimal solution path $\pi\langle s = n_0, n_1, n_2, \ldots, n_k = t\rangle$ can be decomposed into a* prefix *and a* suffix, *via a sidetrack edge $e_{n_{i-1},n_i} \in \pi$, for any i with $0 < i \leq k$ as follows: Let n_i be the* first *node in π which verifies that $g^*(n_i) < g^*(n_{i-1}) + \omega(n_{i-1}, n_i)$, then:*

- *$\phi\langle s = n_0, n_1, \ldots, n_{i-1}\rangle$ is the* prefix, *possibly empty.*
- *$\sigma\langle n_i, n_{i+1}, \ldots, n_k = t\rangle$ is the* suffix, *possibly empty.*
- *The edge $e_{n_{i-1},n_i} \in \pi$ is a sidetrack edge.*

Figure 1 shows a graph with two sidetrack edges. In particular, $e_{C,B}$ decomposes the suboptimal path $\langle s, C, B, t\rangle$ into its prefix $\phi\langle s, C\rangle$ and its suffix $\sigma\langle B, t\rangle$ because $g^*(C) = 3$ and $g^*(B) = 3 < g^*(C) + \omega(C, B) = 5$. The other sidetrack exemplifies the case of sidetrack edges that get to the goal, t. Let π denote any solution path (either optimal or suboptimal) such that $g(n_i) = g^*(n_i), 1 \leq i < k$:

- If $g(t) > g^*(t)$ then π is a suboptimal path, decomposed into its prefix and suffix via the sidetrack $e_{n_{k-1},t}$, because the first node verifying $g^*(n_i) < g(n_{i-1}) + \omega(n_{i-1}, n_i)$ is t indeed.

  Figure 1 shows this case: the edge $e_{E,t}$ decomposes the suboptimal path $\langle s, D, E, t\rangle$ into the prefix $\phi\langle s, D, E\rangle$ and an empty suffix $\sigma = \lambda$.

- If $g(t) = g^*(t)$ then π is an optimal path, to be denoted as π^*, and thus, there is no vertex n_i verifying that $g^*(n_i) < g^*(n_{i-1}) + \omega(n_{i-1}, n_i)$.

  Figure 1 shows this case: the path $\langle s, A, B, t\rangle$ is optimal. Consequently, $g(n_i) = g^*(n_i) = g^*(n_{i-1}) + \omega(n_{i-1}, n_i), 1 \leq i \leq k$ so there is no sidetrack edge. It is then said that the prefix of π is π itself.

These two types of paths are both considered *direct*:

Definition 4. *A path $\pi\langle s = n_0, n_1, n_2, \ldots, n_k = t\rangle$ is said to be* direct *if and only if one of the following conditions hold:*

- *It is an optimal path or, in other words, it has no sidetrack edge.*
- *It is a suboptimal path with an empty suffix, $\sigma = \lambda$.*

and is called indirect *otherwise.*

Note there are also paths with an empty prefix, $\phi = \lambda$: if there is a suboptimal path to get to t from s which consists of only one edge, then it is a sidetrack edge which makes $\phi = \sigma = \lambda$; any suboptimal path $\pi\langle s = n_0, n_1, n_2, \ldots, n_k = t\rangle$ with $g^*(n_1) < \omega(s, n_1)$ is decomposed into an empty prefix, $\phi = \lambda$, and the suffix $\sigma\langle n_1, n_2, \ldots, n_k = t\rangle$ via the sidetrack e_{s,n_1}.

Next, we provide a novel result regarding the relationship between suboptimal solution paths and adjacent paths via a sidetrack edge:

Lemma 1. *Let $\pi_i \in \Pi$ denote a suboptimal solution path, and let $e_{u,v}$ denote its first sidetrack edge, then there exists another solution path $\pi_j \in \Pi, C(\pi_j) < C(\pi_i)$, such that the ending vertex of $e_{u,v}$, v, belongs to the prefix of π_j.*

Proof: From the definition of a sidetrack edge it follows that there is a shorter path to v. Let ϕ denote it. Denoting the subpath from v to the end of π_i by σ, the concatenation of ϕ and σ yields another solution path, π_j which is necessarily shorter than π_i, i.e., $C(\pi_j) < C(\pi_i)$. Note that this is true even if π_i is a direct suboptimal path, i.e., $v = t$. □

3 Centroids

A new definition, which refines the notion of sidetrack edge is proposed first:

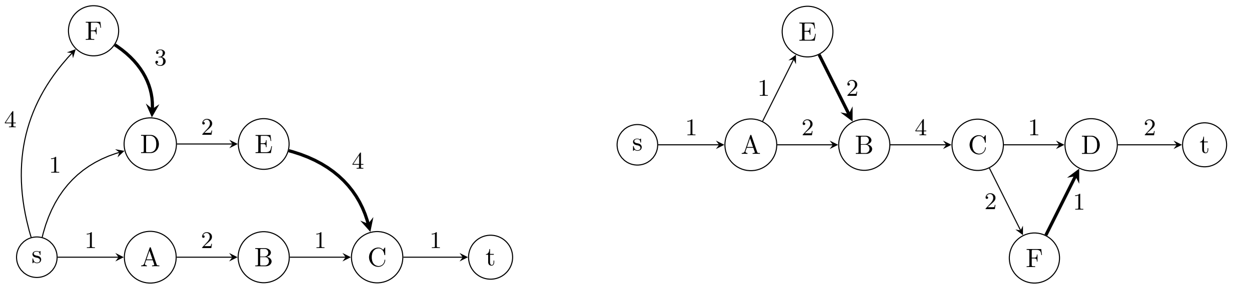
Definition 5. *A* centroid *z is defined as the association of a sidetrack edge $e_{u,v} \in E$ with $u, v \in V$, and an overall cost C_z.*

Hence, two centroids are different if they use different sidetrack edges or they have different overall costs. Clearly, every suboptimal solution path π using a centroid is divided into a prefix and suffix, and $C(\pi) = g^*(u) + \omega(u, v) + g_b(v) = C_z$. Of course, question is how to find the paths defined by the cost and sidetrack edge of the centroid. To do this, we must enumerate all valid suffixes and prefixes for paths of the centroid. Before returning to this question it is first shown that centroids create an *equivalence class* over the set of all suboptimal solution paths, G'_π.

Lemma 2. *Any suboptimal path π is represented by one and only one centroid z.*

Proof: Indeed, there is a unique combination of an overall cost and a sidetrack edge that represents any suboptimal solution path π: It is trivially observed that the cost of π is unique, $C(\pi)$ and thus, its centroid has to have an overall cost $C_z = C(\pi)$; secondly, Definition (3) explicitly uses the *first* sidetrack in π to split the path into its prefix and suffix, and hence it has to be unique. To conclude the second observation, note that every suboptimal path must necessarily have at least one sidetrack, otherwise it would be an optimal path. □

Lemma 3. *The equivalence class induced by the definition of centroids forms a partition over the set of all suboptimal solution paths G'_π, i.e., every suboptimal solution path $\pi_i \in \Pi$ belongs to one and only one equivalence class defined by a centroid z.*



(a) Composition of sidetrack edges

(b) Juxtaposition of sidetrack edges

Figure 2: Example of suboptimal solution paths with various sidetrack edges

As a consequence of the preceding Lemma, the computation of all suboptimal solution paths[2] in the solution set Π can be computed from $\bigcup_{z \in \boldsymbol{Z}} [\![z]\!]$ where $\boldsymbol{Z}$ is the set of all centroids of our problem with cost less than or equal to C^*_φ, and $[\![z]\!]$ is the set of all paths that use centroid z. We show next that a unique centroid z represents an arbitrary number of suboptimal solution paths $[\![z]\!]$ that can each contain a different number of sidetrack edges. The only possible cases are shown in Figure 2. Figure 2a shows the case where the sidetrack edges of centroids (shown with thick lines) can be *composed* to create suboptimal solution paths which are larger. In particular, the cost of the path $\langle s, D, E, C, t\rangle$, 8, is smaller than the cost of the solution path $\langle s, F, D, E, C, t\rangle$, 14, which results of the composition of the sidetrack edge defining its centroid $\langle e_{F,D}, 14\rangle$ with the defining sidetrack edge of the centroid representing the former, $\langle e_{E,C}, 8\rangle$. Figure 2b shows a more interesting case, where the sidetrack edges of two centroids (shown with thick lines) can be *juxtaposed* so that taking them creates a suboptimal solution path which is larger than the cost of any suboptimal solution path that uses only one of the two edges. To be clear, the solution set Π with $\kappa = 4$ is shown next:

$$\begin{array}{llll} \pi_0: & \langle s, A, B, C, D, t\rangle & C^*_0: \ 10 & - \\ \pi_1: & \langle s, A, E, B, C, D, t\rangle & C^*_1: \ 11 & \langle e_{E,B}, 11\rangle \\ \pi_2: & \langle s, A, B, C, F, D, t\rangle & C^*_2: \ 12 & \langle e_{F,D}, 12\rangle \\ \pi_3: & \langle s, A, E, B, C, F, D, t\rangle & C^*_3: \ 13 & \langle e_{E,B}, 13\rangle \end{array}$$

Restricting attention only to the suboptimal paths, π_1 uses the sidetrack edge $e_{E,B}$ and has an overall cost $C^*_1 = 11$, so that the centroid $\langle e_{E,B}, 11\rangle$ represents it. Note that this sidetrack splits π_1 into its prefix $\phi = \langle s, A, E\rangle$ and its suffix $\sigma = \langle B, C, D, t\rangle$. Because, there is a path from B to t with cost 7, $g_b(B) = 7$, then the overall cost of the path can be computed as $g^*(E) + \omega(E, B) + g_b(B) = 2 + 2 + 7 = 11$. However, π_3 contains both sidetracks $e_{E,B}$ and $e_{F,D}$. According to Definition (3), it is the first sidetrack edge in π_3, $e_{E,B}$ which splits π_3 into its prefix $\phi = \langle s, A, E\rangle$ and its suffix $\sigma = \langle B, C, F, D, t\rangle$. The cost of the suffix is the g_b cost of node B for constructing π_3, which is equal to $g_b(B) = 9$, hence $C^*_3 = C(\pi_3) = g^*(E) + \omega(E, B) + g_b(B) = 2 + 2 + 9 = 13$. As a result, the node B has two g_b costs, 7 and 9 used respectively for π_1 and π_3. In conclusion, the juxtaposition of several sidetrack edges in the same suboptimal solution path is represented with different g_b values in the same node. The last column above shows the centroid of each suboptimal path, where it can be seen that the sidetrack $e_{E,B}$ has been used twice to generate π_1 and π_3.

To conclude, any suboptimal solution path results from either the consideration of solely the defining sidetrack edge of a centroid, or the combination of an arbitrary number of them, either composed, juxtaposed or a combination of both.

4 BELA$_0$

We consider first the uninformed variant of our search algorithm, BELA$_0$, where heuristics are not available. From the preceding Section, the computation of the κ shortest-paths can be computed from the union of all centroids with cost less than or equal to C^*_φ, where every centroid is defined as the association of a sidetrack edge and an overall cost. As indicated in the Definitions, C^*_0 is the cost of all the optimal solution paths and an ordinary application of Dijkstra's can be used to compute all of them. For the case of a centroid z such that $C_z = C^*_i$, $i \geq 1$, we will soon show how to compute its set of paths from its cost and sidetrack edge.

The first extension that we propose to Dijkstra's algorithm consists of storing all edges traversed in the CLOSED list. When a duplicate is found (e.g., node D in Figure 2a), the edge to it (i.e., $e_{F,D}$) is stored in CLOSED, and the node is not re-expanded. This way, all existing sidetrack edges can be easily distinguished from tree edges: Given a node n in CLOSED, one of its incoming edges $e_{m,n}$ is a sidetrack edge if and only if $g^*(n) < g^*(m) + \omega(m, n)$. This operation can be performed in $O(1)$ because Dijkstra's algorithm already stores in CLOSED the optimal cost to

[2]Note that the solution set Π is not required to contain all suboptimal solution paths with the last cost, C^*_φ.

reach each node from the start state, $g^*(\cdot)$, and so does BELA_0. According to the normal operation of Dijkstra's, when the goal state is about to be expanded it knows that a new direct (either optimal or suboptimal) path has been found. Because it also knows the cost of the new path and its parent is already in CLOSED, it can output the new solution path by following all backpointers. Conducting a depth-first search in CLOSED where the next node is a parent of the current one with a g^*-value equal to the g^*-value of the current node minus the cost of the edge that joins them, delivers all the direct solution paths with a cost equal to the desired one. We call this process *prefix construction*.

More specifically, prefix construction, shown in Algorithm 1, is the process of computing all optimal paths from the start state to a designated node in CLOSED by following the backpointers from it, and also discovering new centroids if any exist. When using a centroid $\langle e_{u,v}, C_z\rangle$ only one specific g_b-value is used among all in the ending vertex, $g_b = C_z - g^*(u) - \omega(u,v)$. Thus, the g_b-value of any node selected when enumerating a prefix can be computed as the sum of the g_b-value of its descendant plus the cost of the edge as shown in Lines 1–2 of Algorithm 1. Moreover, once a new g_b-value is discovered, the existence of new centroids can be verified by checking the condition given in Definition (2), see Line 4. Finally, the enumeration of prefixes is done recursively in Lines 8–10, where $\otimes$ denotes the cross-product of its arguments.

Algorithm 1: Pseudocode of `GetPrefixes`

Data: A node n and a backward g-cost, g_b
Result: All optimal paths from s to n. In addition, it creates new centroids if any is found.

1 **if** $g_b \notin g_b(n)$ **then**
2   `insert` $(g_b(n),\ g_b)$
3   **for** *every parent* p_n *of node* n **do**
4   **if** $g^*(n) < g^*(p_n) + \omega(p_n, n)$ **then**
5   `insert(Z`, $g^*(p_n) + \omega(p_n, n) + g_b$`)`
6 **if** $n = s$ **then**
7   return $\{s\}$
8 **for** *every parent* p_n *of node* n **do**
9   **if** $g^*(p_n) + \omega(p_n, n) = g^*(n)$ **then**
10         $\phi \leftarrow \phi \cup$ `GetPrefixes(`$p_n,\ g_b + \omega(p_n, n)$`)`$\otimes\{n\}$;
11 return ϕ

Given a centroid z, i.e., a sidetrack edge and a known overall cost C_z, all solution paths that go from s to the starting vertex of the sidetrack edge, traverse it and then arrive at the goal state with overall cost C_z, or simply, the paths of the centroid, are computed as the cross-product of all prefixes and suffixes corresponding to the centroid. As in the case of prefix construction, all suffixes with a given cost can be found by conducting a depth-first search in CLOSED where the next node selected is a child of the current one with a g_b-value equal to the g_b-value of its parent minus the cost of the edge that joins them, until the goal state is reached. This procedure is known as *suffix construction*.

Algorithm 2 shows the pseudocode of $\text{BELA}_0/\text{BELA}^*$, where $\boldsymbol{Z}$ is the set of the current centroids and p_n represents the parent of a node n. The function `GetPaths` computes all paths represented by the centroid given to it as the cross-product of all its prefixes and suffixes. To guarantee *admissibility* the algorithm first checks whether there is a centroid with a cost strictly less or equal than the current $f(n)$ value, with n being the node just popped out from OPEN. If so, all its paths are added to the solution path until no more centroids can be used, or κ shortest paths have been found, in which case the algorithm returns. Otherwise, if a direct path to the goal has been found, then a new centroid with the cost of this solution is added and the current iteration is skipped. Because Dijkstra expands nodes in ascending order of cost, this guarantees that the next iteration will start by outputting all paths corresponding to the centroid just added. In case the current node, n, has been already expanded, its edge is added to the CLOSED list and, before skipping the current iteration, it is verified whether there are known g_b-values of it in CLOSED. If so, this node has known suffixes, therefore, new centroids have been discovered, so we add them to the set of centroids, $\boldsymbol{Z}$. Finally, the current node is expanded and its children are added to the OPEN list in increasing order of their f-value.

Note that the OPEN list can be exhausted without having found κ shortest paths. In such case, the algorithm considers all centroids in increasing order of their cost adding their solution paths to Π. While computing these paths, new centroids might be discovered and so the loop proceeds until κ paths have been found. If after considering all centroids, κ paths are not found, the algorithm simply returns those that were found.

It is noted that Algorithm 2 follows the same mechanics as Dijkstra's/A*, the only difference being that it uses information in CLOSED to reconstruct the κ paths and, for this, it uses an ordered set of centroids to generate paths from, $\boldsymbol{Z}$. Hence, its theoretical properties naturally derive from those of Dijkstra's/A*.

Lemma 4 (Sufficient condition for expansion)**.** *$BELA_0$ expands all nodes with $f(n) < C^*_\varphi$.*

Algorithm 2: Pseudocode of $BELA_0/BELA^*$

Data: A graph $G(V, E)$ and two designated vertices $s, t \in V$
Result: Solution set Π with κ shortest paths

1 closed $\leftarrow \varnothing$, Z $\leftarrow \varnothing$
2 open $\leftarrow \{s\}$ with $g(s) = 0$
3 **while** open $\neq \varnothing$ **do**
4     $n \leftarrow$`pop(open)`
5   **while** $\exists z \in$Z, $C_z \leq f(n)$ **do**
6         $z \leftarrow$`pop (Z)`
7         $\Pi = \Pi \cup$`GetPaths` (z)
8   **if** $|\Pi| \geq \kappa$ **then**
9   return Π
10   **if** $n = t$ **then**
11   `insert(Z,` $\langle e_{p_t,t}, g(n)\rangle$`)` in increasing order of cost
12   continue
13   **if** $n \in$closed **then**
14   `add` (closed, $e_{p_n,n}$)
15   **for** *every g_b-value in n* **do**
16   `insert(Z,` $\langle e_{p_n,n}, g^*(p_n) + \omega(p_n, n) + g_b\rangle$`)` in increasing order of cost
17   continue
18   `insert(closed,` n`)`
19   **for** $c_i \in$`children(`n`)` **do**
20         $g(c_i) = g(n) + \omega(n, c_i)$
21   `insert(open,` c_i`)` in increasing order of $f(\cdot)$
22 **while** Z $\neq \varnothing$ **do**
23     $z \leftarrow$`pop (Z)`
24     $\Pi = \Pi \cup$`GetPaths` (z)
25   **if** $|\Pi| \geq \kappa$ **then**
26   return Π
27 return Π

Proof: Once $BELA_0$ discovers a centroid, it is stored in $\boldsymbol{Z}$ for consideration only once the $f(n)$ value of the current node from OPEN is greater or equal than the cost of the cheapest centroid in $\boldsymbol{Z}$ —see Line 5 in Algorithm 2. Because $f(n) = g(n)$ is monotonically increasing and nodes from OPEN are expanded in ascending order of $f(n)$, the κ paths can be discovered only once $C^*_\varphi \leq f(n)$, thus, after expanding all nodes with $f(n) < C^*_\varphi$. □

Lemma 5 (Necessary condition for expansion)**.** *$BELA_0$ never expands nodes with $f(n) > C^*_\varphi$. Thus, a necessary condition for expansion in $BELA_0$ is $f(n) \leq C^*_\varphi$.*

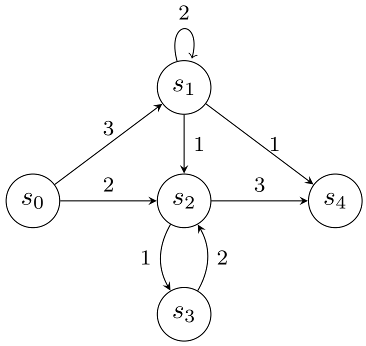
Proof: Once a centroid z is considered and κ paths are generated, the algorithm halts execution —see Line 9. From the proof of the preceding Lemma, it is observed that this happens as soon as $C^*_\varphi \leq f(n)$ and the considered centroid yields the necessary number of paths to complete the search of κ shortest paths, i.e., no node with $f(n) > C^*_\varphi$ is ever expanded. □

Theorem 1 (Commpleteness and Admissibility)**.** *$BELA_0$ finds all paths in Π as given in Definition (1).*

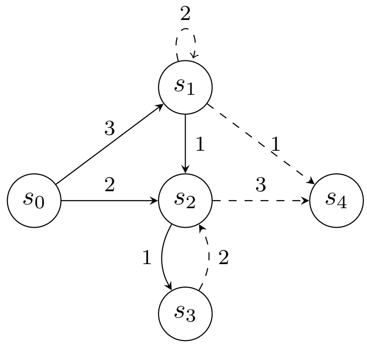
Proof: Let π_i denote a path in Π, and let us consider two different cases: Either π_i is a direct path or it is indirect —see Definition (4).

If it is a direct path, its node corresponding to t will eventually be expanded, and thus inserted in $\boldsymbol{Z}$ in Line 11. Because nodes in OPEN are considered in increasing order of cost, the next node to expand will have a cost equal or greater than the cost of π_i and thus, all paths represented by the latest centroid will be discovered in the loop of Lines 5–9.

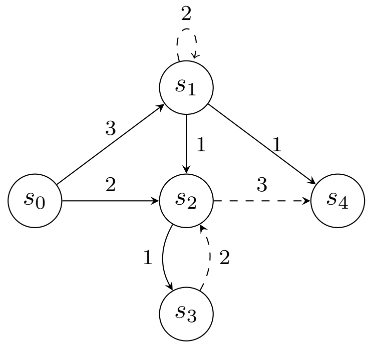
If it is an indirect path, then it must contain at least one sidetrack edge, see Definition (3). Let $e_{u,v}$ denote its first sidetrack edge and let C_z denote its cost. There are two cases to consider: Either the ending vertex of the sidetrack edge, v, has at least one g_b-value when reached from u, or it does not. In the case it has at least one g_b-value, then a new centroid z, with $C_z = g^*(u) + \omega(u, v) + g_b(v)$, is created for every g_b-value of v in Line 16. Once the centroid representing π_i is created, it will be eventually considered and π_i found as soon as the first node n with $f(n) \geq C_z$ is popped out from OPEN. Note that this has to happen because π_i is assumed to have a cost strictly less or equal than C^*_φ, and the sufficient and necessary conditions for expansion guarantee that a node n with $f(n) \geq C^*_\varphi \geq C_z$ should be eventually popped out from OPEN. In case that vertex v has no g_b-values when reached from u, then according to Lemma (1) there shall exist a path $\pi_j \in \Pi$, such that $C(\pi_j) \leq C(\pi_i)$, and v belongs to the prefix of π_j. Upon discovery of the path π_j, the prefix construction procedure necessarily will set a g_b-value for vertex v and it will discover $e_{u,v}$, and thus a centroid z will be created that represents π_i. As in the previous case, this centroid will be eventually considered and π_i will be found. □



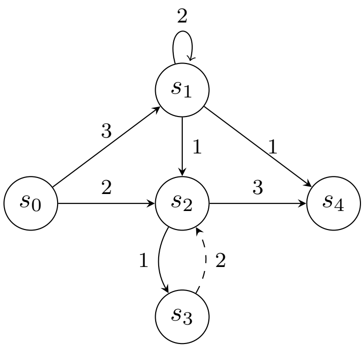
(a) Simple example of BELA_0. s_0 is the start vertex and s_4 is the goal vertex.



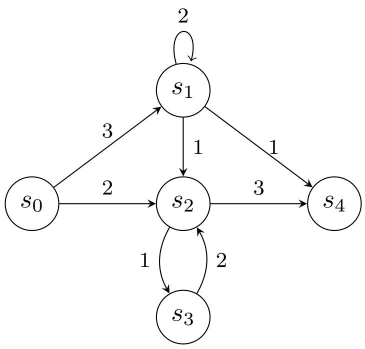
(b) The result of the first five iterations of BELA_0. The solid lines represent edges stored in CLOSED, whereas dotted lines represent edges whose start nodes have only been generated, but not expanded and are therefore not in CLOSED.



(c) Result of the fifth iteration of BELA_0. s_4 is chosen for expansion next, at the beginning of the sixth iteration



(d) Beginning of the ninth iteration of BELA_0.



(e) End of the ninth iteration of BELA_0. All nodes have been expanded and OPEN is empty

Figure 3: Example of BELA_0

The following result ensures that BELA_0 preserves the best known asymptotic worst-case complexity:

Theorem 2 (Algorithmic complexity). *$BELA_0$ runs in $O(|E| + |V| \log |V| + \kappa|V|)$.*

Proof: Algorithm 2 has a complexity at least as bad as Dijkstra's, $O(|E|+|V|\log|V|)$(Dechter & Pearl, 1985). On top of this, it has to output κ paths in explicit form, which come from the cross-product of all prefixes and suffixes of each centroid. In the worst case every centroid yields a single prefix and suffix, and thus, the added complexity is $O(\kappa|V|)$, with $|V|$ being the length of the shortest paths in the worst case, resulting in a worst-case time complexity of $O(|E| + |V| \log |V| + \kappa|V|)$. □

To conclude the presentation of the brute-force variant of our algorithm, the first example introduced in the description of K*(Aljazzar, 2009; Aljazzar & Leue, 2011) is solved using BELA_0. The example considered is shown in Figure 3, where it is requested to find three shortest paths between s_0 and s_4 in the graph shown in Figure 3a, i.e., $\kappa = 3$. The algorithm shown in Pseudocode 2 is considered next using $f(n) = g(n)$, i.e., ignoring any heuristic guidance.

Figure 3b shows the first five iterations of BELA_0. Because no centroids have been discovered yet, the algorithm proceeds in exactly the same fashion as Dijkstra's. The expansion order (with the g^*-values shown between parenthesis) is $s_0(g = 0)$, $s_2(g = 2)$, $s_1(g = 3)$, $s_3(g = 3)$, and $s_2(g = 4)$. Here, ties are broken favoring nodes which enter OPEN earlier. A solid line in Figure 3b indicates that the start vertex of the edge has been expanded, and thus, it is present in CLOSED. At this point, the contents of OPEN (with the g-values shown between parenthesis) are: $s_4(g = 4)$, $s_4(g = 5)$, $s_1(g = 5)$, $s_2(g = 5)$. The dotted lines are incoming edges of nodes that have only been generated, but not expanded, and are thus only in OPEN and not CLOSED.

In the sixth iteration, $s_4(g = 4)$ is popped out from OPEN, which is detected to be the goal on Line 10 of Algorithm 2. As a consequence, the first centroid, $\langle e_{s_1,s_4}, 4\rangle$ is added to the set of centroids $\boldsymbol{Z}$, and the current iteration is skipped without generating any children.

Next, the node $s_4(g = 5)$ is popped from OPEN, but before it is examined it is observed that there is a centroid with cost 4, being less or equal than the g-value of the current node, 5, on Line 5. Thus, the centroid $\langle e_{s_1,s_4}, 4\rangle$ is popped from $\boldsymbol{Z}$ (so that it becomes empty) and all paths represented by this centroid are computed by `GetPaths`. The prefixes of the starting vertex of the centroid, s_1 are all the optimal paths from the start state, s_0 to it. There is only one such prefix, namely $\langle s_0, s_1\rangle$ with a cost equal to 3. Recall that while prefixes are computed, all visited nodes are checked for incoming sidetrack edges, which give us new centroids. Currently, s_1 only has one incoming edge which is a tree edge, and s_0 has no incoming edges, so no new centroids are found. Because the cost of the prefix plus the cost of the defining edge of the centroid is equal to the overall cost of the centroid, 4, there are no suffixes to compute. Therefore, the first path found is $\pi_1 : \langle s_0, s_1, s_4\rangle$.

After examining all paths represented by the centroid considered, the next iteration proceeds as usual. Next, $s_4(g = 5)$ is popped from open OPEN. s_4 is found to represent the goal vertex on Line 10, so the centroid $\langle e_{s_2,s_4}, 5\rangle$ is added to $\boldsymbol{Z}$, and the current iteration is terminated without expanding s_4.

At the beginning of the eighth iteration, the contents of CLOSED are still the same as those shown in Figure 3c, but the OPEN list has shrunk. Now, it only contains $s_1(g = 5)$ and $s_2(g = 5)$. Thus, the next node chosen for expansion on Line 4 is s_1. However, before proceeding, it is observed that there is currently a centroid, $\langle e_{s_2,s_4}, 5\rangle$ with a cost less than or equal to the g-value of the node just popped from OPEN, so GETPATHS is invoked again on this centroid.

GETPATHS starts by assigning a g_b-value of 3 to node s_2, because that is the cost of the centroid under consideration minus the cost of the prefix. It then starts computing the prefixes from the starting vertex of the centroid, s_2. First, it discovers an incoming sidetrack edge of s_2, e_{s_1,s_2}. Because s_2 has a g_b-value 3, the g^*-value of s_1 is 3, and the edge cost of the sidetrack edge is 1, a new centroid, $\langle e_{s_1,s_2}, 7\rangle$ is added to $\boldsymbol{Z}$ which now only contains this centroid. It then continues following backpointers, reaching the start state, which has no incoming edges. Thus, the full list of prefixes just contains $\langle s_0, s_2\rangle$. Like earlier, the list of suffixes is empty because the ending vertex of the centroid currently under consideration is the goal state itself, and thus has a g_b-value equal to 0. Hence, the only path returned is $\pi_2 : \langle s_0, s_2, s_4\rangle$.

We still have not returned $\kappa = 3$ paths, so we continue with the eighth iteration. The current node, s_1 is already in CLOSED, so the new edge (the self-loop, e_{s_1,s_1}) is added to CLOSED on Line 14. Before continuing to the next iteration, it is observed that s_1 contains one g_b-value, 1, and thus, a new centroid $\langle e_{s_1,s_1}, 6\rangle$ is added to $\boldsymbol{Z}$. The cost of the new centroid, 6, is computed as the sum of the g^*-value of node s_1, plus the cost of the sidetrack edge, 2, plus the g_b-value, 1. Currently, the list of centroids stored is: $\langle e_{s_1,s_1}, 6\rangle$, $\langle e_{s_1,s_2}, 7\rangle$. Because the node popped out from OPEN was a duplicate, the current iteration is terminated before expansion.

In the ninth iteration, there is only one node in OPEN, $s_2(g = 5)$, and the CLOSED list has been updated to contain the self-loop of node s_1 as shown in Figure 3d. At this point, $\boldsymbol{Z}$ contains two centroids, but none of them have a cost less than or equal to the g-value of the node popped from OPEN, so they are ignored. Because s_2 is not the goal state, it is looked up in CLOSED and it is found to be a duplicate, so the edge e_{s_3,s_2} is added to CLOSED. Recall that node s_2 had a known g_b-value equal to 3, and thus the loop on lines 19–21 adds a new centroid, $\langle e_{s_3,s_2}, 8\rangle$, where the cost of the centroid is computed as follows: $g^*(s_3)+\omega(s_3, s_2)+g_b(s_2) = 3+2+3 = 8$. Currently, $\boldsymbol{Z}$=$\{\langle e_{s_1,s_1}, 6\rangle$, $\langle e_{s_1,s_2}, 7\rangle$, $\langle e_{s_3,s_2}, 8\rangle\}$. Because this node has been found in CLOSED the current iteration ends. The result of this iteration is shown in Figure 3e

At the end of the ninth iteration, the OPEN list is empty, meaning all nodes have already been expanded, so execution continues in the loop starting at Line 22. From this point on, BELA$_0$ only uses information from the CLOSED list to find new paths. Note it can update CLOSED with new centroids if any are found during prefix computation.

The first centroid popped is $\langle e_{s_1,s_1}, 6\rangle$. Exactly as it happened in the eighth iteration, the prefix computation starts by setting a new g_b-value of 3 for the starting vertex of this centroid, s_1, equal to the cost of the current centroid, 6, minus the cost of the prefix, 3. While computing the prefixes from s_1 it realizes (again) that the self-loop e_{s_1,s_1} is a sidetrack edge, as it did in the eighth iteration, so a new centroid $\langle e_{s_1,s_1}, 8\rangle$ is added to $\boldsymbol{Z}$ with the cost of the centroid computed like so: $g^*(s_1)+\omega(s_1, s_1)+g_b(s_1) = 3+2+3 = 8$. The next node visited is s_0, which has no incoming edges, so the prefix computation ends, returning the path $\langle s_0, s_1\rangle$. This time, the suffix computation produces a non-empty path. Starting at s_1 with a g_b-value equal to 1, it chooses all descendants n' with a g_b-value equal to $g_b(s_1) - \omega(s_1, n')$. There is only one such descendant, s_4, so the only suffix produced is $\langle s_1, s_4\rangle$. Finally, `GetPaths` computes the cross-product of all prefixes and suffixes giving us the path $\pi_3 : \langle s_0, s_1, s_1, s_4\rangle$, with a cost equal to 6. Because the number of paths produced so far, 3, is equal to the desired number of paths, $\kappa = 3$, the algorithm terminates on Line 26, returning the following paths:

$$\begin{array}{lll} \pi_1 & \langle s_0, s_1, s_4\rangle & C_0^* = 4 \\ \pi_2 & \langle s_0, s_2, s_4\rangle & C_1^* = 5 \\ \pi_3 & \langle s_0, s_1, s_1, s_4\rangle & C_2^* = 6 \end{array}$$

5 BELA*

This Section considers the availability of a heuristic function, $h(\cdot)$, which is assumed to be *consistent* and thus, *admissible*. As a matter of fact, all of the discussion from the previous Section apply to this one and only a few novel remarks are necessary. Indeed, Algorithm 2 becomes BELA* when using $f(n) = g(n) + h(n)$.

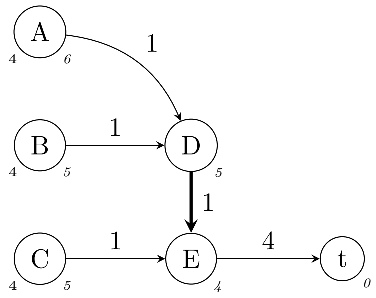


Figure 4: Expansion order of BELA*

The first observation is that $f(\cdot)$ is monotonically increasing, provided that $h(\cdot)$ is consistent, as assumed. From this, Lemmas (4) and (5) are still valid for BELA*.

The case of Theorem (1) deserves further consideration, though. Consider Figure 4, where the g-value of each node is shown below it to the left, and its h-value appears below it to the right in italics. Consider next the order expansion of BELA_0 and how it contrasts with the order expansion of BELA*. BELA_0 first expands node C (g^*=4), generating node E; next it expands nodes B ($g^* = 4$) and A ($g^* = 4$), which generate two copies of node D with $g^* = 5$. After expanding node E ($g^* = 5$), the goal is generated with a cost equal to 9. The first copy of node D is expanded immediately after, adding a new copy of node E ($g^* = 6$) to the OPEN list. The next node in open is the second copy of node D ($g^* = 5$) which is found to be a duplicate and thus, it is skipped for expansion after adding the edge $\langle A, D\rangle$ to the CLOSED list. The next node popped from OPEN is the node E ($g^* = 6$) which, as in the previous case, is found to be a duplicate so that the edge $\langle D, E\rangle$ is added to CLOSED. The last node in OPEN is t, which generates a new centroid with a cost equal to 9. In the next iteration, the centroid just created is considered and, while constructing its prefixes, node E gets a g_b-value equal to 4 which is propagated backwards from the goal as the sum of all cost edges traversed so far. At this point, the prefix construction procedure notes that there is an incoming sidetrack edge, $\langle D, E\rangle$, and hence a new centroid with cost $C = g^*(D) + \omega(D, E) + g_b = 5 + 1 + 4 = 10$ is added. The prefix construction will continue moving backwards to C and from there eventually reaching the start state s, setting g_b values for all nodes traversed. The important aspect to note is that, if κ nodes have not been produced yet, BELA_0 will consider the centroid $\langle e_{D,E}, 10\rangle$ created in the last prefix construction. Proceeding as in the previous case, it will search both prefixes starting from node D, one through A and the other one through B. As it can be seen, BELA_0 provides a *one to many* mapping of centroids to solution paths, in contrast to K* which provides a *one to one* mapping from paths in the *path graph* and paths in the true graph.

Still, the same observation is true for BELA*, but the cardinality of this mapping might shrink as a result of *tie-breaking* rules of f-values. When using BELA* nodes are expanded in increasing order of their f-value. Succinctly, C ($f = 9$) and B ($f = 9$) are expanded first generating E ($f = 9$) and D ($f = 10$), respectively. After expanding E ($f = 9$), the goal is generated with cost 9 which becomes the first node in OPEN, so it is selected for expansion in the next iteration, triggering the prefix construction, which returns all optimal paths from s to t with a cost equal to 9 that use the edge $\langle E, t\rangle$. Note, however, that node D is still in OPEN and thus, the sidetrack edge $\langle e_{D,E}, 10\rangle$ has not been discovered yet, as with BELA_0. There are two nodes in OPEN at this point, A ($f = 10$) and D ($f = 10$). The expansion order matters indeed and if A is expanded before D, paths will be discovered in the same order as BELA_0. Assuming the opposite, the expansion of D ($f = 10$) generates E ($f = 10$), which is found to be a duplicate, and the edge $\langle D, E\rangle$ is added to CLOSED. This time, before skipping its expansion, Lines 15–16 in Algorithm 2 add a new centroid because node E was already given a g_b-value equal to 4. Because the new centroid has a cost equal to 10 units, which is equal to the f-value of the next node in OPEN, it is considered before expanding node A. Because A has not yet been expanded, the incoming edge $\langle A, D\rangle$ is not discovered in prefix construction. BELA* will have to wait for the expansion of node A to realize that a new centroid can be created and thus, the existence of new paths. Consequently, only one prefix will be considered even though there are two.

There are two important consequences of the expansion order of BELA*: On one hand, the consideration of a centroid might yield less paths than the number of paths returned by BELA_0 when considering the same centroid; On the other hand, and only in the context of BELA*, centroids can be constructed from a tree edge! Note that in the last example, the eventual expansion of node A ($f = 10$) generates D ($f = 10$) which, when being expanded is observed to have a g_b-value equal to 4, and thus a new centroid is generated. However, the edge $\langle A, D\rangle$ is a tree edge. There remains a third consequence. In spite of the effects of *tie-breaking* policies on the expansion order of BELA*, all cases considered in the Proof of Theorem (1) are still valid and thus, BELA* is both complete and admissible. Moreover, like A* when contrasted with Dijkstra's, BELA* should considerably reduce the number of expansions in comparison with BELA_0 by avoiding the consideration of all nodes with $f(n) > C^*_\varphi$. The accuracy of the heuristic function plays a major role in the level of reduction. The better the heuristic, the larger the improvement in the number of necessary expansions.

To conclude, note that when using a consistent heuristic, duplicates are never expanded, thus Theorem (2) still applies.

6 Empirical evaluation

This last section provides all relevant details of the experiments described in the main paper. All of the source code, along with documentation, unit tests, and various scripts for running the experiments and generating figures and tables are available on github[3]. All the instances for all experiments are stored in Zenodo (Linares López & Herman, 2024). The selection of domains considers both map-like and combinatorial domains, with branching factors ranging from slightly above 2 (in the roadmap domain), to two-digit branching factors in the N-Pancake domain; depths ranging from dozens of vertices (as in the N-Puzzle or the N-Pancake domains) to several hundreds, often exceeding 1,000 —as in the Random Maps and the Roadmap domains. We also consider both unit cost and non-unit cost versions (the definition of non-unit costs is domain dependent). The selection of κ values has been always from 1 to 10, from 10 to 100 in steps of 10, next getting to 1,000 in steps of 100 and, finally, to 10,000 in steps of 1,000, unless inferior values were enough to compare the selected algorithms, or too hard to solve. The benchmarking suite has been configured so that every algorithm is able to solve all instances for all the selected values of κ.

In each domain, we measure runtime, number of expansions, and memory usage for each algorithm. Importantly, memory usage is simply the memory measured at the termination of the algorithm, with the memory needed for storing solutions subtracted. Data is provided first, as figures, and also in tabular form in Appendix A.

All the experiments have been executed on a machine with 8 core i7 and 32 Gb of RAM. All algorithms have been implemented in c++-17.

6.1 Roadmap

The roadmap domain was used in the empirical evaluation of K* in (Aljazzar & Leue, 2011) and thus, it is considered in this section. It is taken from the 9th DIMACS Shortest-Path Challenge. Two variants are considered, dimacs and unit. The first uses the provided edge costs. The latter considers all edges to have cost 1.

6.1.1 9th DIMACS Challenge

Figures 5–10 show the results of running BELA*, mA*, K*, and their brute-force variants over a selection of maps from the 9th DIMACS Shortest Paths Challenge. In the empirical evaluation of K* only NY and E were used. Table 1 shows all of the available maps and their size measured in the number of vertices and edges. The figures show the runtime (in seconds), memory usage (in Mbytes) and number of expansions of each algorithm. Every point has been averaged over 100 instances randomly generated where, as in the original evaluation of K* a random pair $s - t$ was accepted if and only if the distance between them was at least 50 km measured as the great-circle distance using the haversine function, as described in (Aljazzar & Leue, 2011).

Name	Description	# Vertices	# Edges
USA	Full USA	23,947,347	58,333,344
CTR	Central USA	14,081,816	34,292,496
W	Western USA	6,262,104	15,248,146
E	Eastern USA	3,598,623	8,778,114
LKS	Great Lakes	2,758,119	6,885,658
CAL	California and Nevada	1,890,815	4,657,742
NE	Northeast USA	1,524,453	3,897,636
NW	Northwest USA	1,207,945	2,840,208
FLA	Florida	1,070,376	2,712,798
COL	Colorado	435,666	1,057,066
BAY	San Francisco Bay Area	321,270	800,172
NY	New York City	264,346	733,846

Table 1: 9th DIMACS Shortest Path Challenge

Figure 5 compares only BELA$_0$, K$_0$ and mDijkstra, and it shows a clear trend. Even if K$_0$ is faster than BELA$_0$ for large values of κ, this only occurs in the smallest graphs, NY and BAY. In larger graphs, the margin of improvement in runtime provided by BELA$_0$ increases with the κ — see Figures 5d and 5f. Note that mDijkstra, the brute-force variant of mA* performs so poorly that it was not practically possible to compute more than $\kappa = 10$ paths with it, while either BELA$_0$ of K$_0$ output 10,000 paths in roughly the same amount of time.

[3]https://github.com/clinaresl/ksearch

Regarding memory usage, Figure 6a shows an effect that will be seen in other experiments as well, i.e., that memory usage in either $BELA_0$ or $BELA^*$ can decrease when increasing the number of paths to seek, κ. This phenomena is attributed to the fact that the number of centroids can decrease when looking for more paths and thus, less memory is required to store all the necessary information, as shown in the example discussed in Section 4.

Figure 8 shows the runtime (in seconds) of $BELA^*$, mA^* and K^*. Again, the same trend we observed before is seen here. Even though K^* performs better than $BELA^*$ in smaller graphs, this margin shrinks as the size of the graph increases, and, eventually, it performs worse in the larger map, E. The relatively good performance of K^* in this domain is attributed to a variety of factors. On one hand, the maximum number of paths requested, 10,000 does not require expanding the whole graph as Figure 10 shows. Second, all of the graphs where K^* performs better than $BELA^*$ are rather small (the largest one with less than 2 million vertices). Third, of all the benchmarks tried, this is the one with the lowest branching factor. Most importantly, the heuristic function suggested in (Aljazzar & Leue, 2011) is very poor. Figure 13 (where mA^* has been removed due to its poor performance) shows that the reduction in the number of expansions is always around 10% both for $BELA^*$ and K^*, which is too small to pay off for the extra work at each node for computing the heuristic function. In fact, the brute-force variants of both K^* and $BELA^*$, i.e., K_0 and $BELA_0$, outperform their heuristic counterparts in all maps, as shown in Figure 11, with the only exception being the smallest graphs, NY and BAY. As a consequence of the poor performance of the heuristic function, mA^* performs the worst, as it expands nodes near the start state many times.

In the end, $BELA_0$ is the fastest among all algorithms tried in this domain in the majority of cases.

6.1.2 Unit variant

The edge costs found in the dimacs variant of the roadmap domain vary quite a lot and are decently large. Weighting every edge with these costs makes the mapping between centroids and solution paths provided by $BELA^*$ to be very poor, because each centroid can only be expected to represent a few paths. For example, $BELA^*$ exploits about 1,800 centroids to generate 10,000 paths in the NY map, whereas in the E map (which is larger), on average, it uses almost 1,100 centroids to generate the same number of solution paths. Thus, every centroid generated in the dimacs variant of this domain approximately represents 5 to 9 paths. Simply using unit costs produces a dramatic change in these figures. Of course, doing so invalidates the heuristic function used in the dimacs variant and thus, only the brute-force versions are considered in this case. In the unit variant of the roadmap domain, $BELA_0$ needs 15 centroids on average to generate 10,000 paths in the NY map, and a little bit more than 18 in the E map to generate the same number of paths, improving the number of paths per centroid by about two orders of magnitude.

The experiments conducted in the unit variant aim to demonstrate how $BELA_0$ can benefit from this increase in the number of paths per centroid. Figure 14 shows the runtime (in seconds) of all brute-force search algorithms in the unit variant. Again, mDijkstra performs so poorly that only $\kappa = 10$ paths can be computed in the time used by K_0 and $BELA_0$ to find 10,000 distinct paths. As Figure 14 shows, the difference in running time between K_0 and $BELA_0$ increases with larger values of κ in all maps, regardless of their size.

Thus, $BELA_0$ strongly dominates both K_0 and mDijkstra in the unit variant of the roadmap domain, being three or four times faster than K_0.

6.2 Random maps

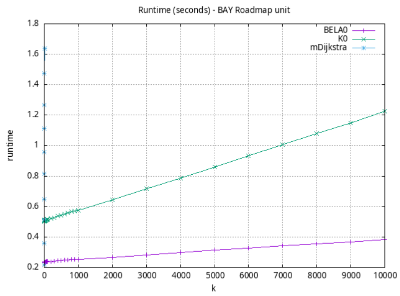
The random map is taken from the 2d Pathfinding *movingai* benchmark[4]. Only the first instance from the random maps benchmark has been used (with 512×512 locations), but considering different percentages of obstruction: 10, 15, 20, 25, 30 and 35, yielding a total of 6 different random maps. For each map, 100 instances were randomly generated where the heuristic distance between the start and goal state is at least 90% of the largest possible distance. All results are averaged over all runs.

6.2.1 Unit variant

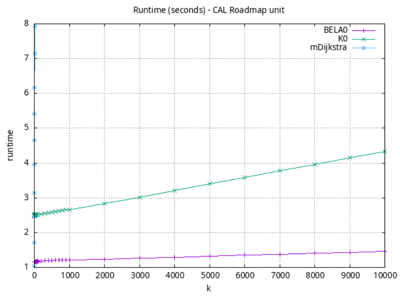
In the first variant it is only possible to move either horizontally or vertically, and the cost of all operators is equal to 1. Both brute-force and heuristic variants of all search algorithms are considered. The heuristic function used is the Manhattan distance.

Figures 17–19 show the runtime (in seconds), the memory usage (in Mbytes), and the number of expansions of $BELA_0$, K_0, and mDijkstra. The first observation is that with the absence of a heuristic function, mDijkstra performs even worse than in the previous domain, and it only finds $\kappa = 4$ paths before using more time than $BELA_0$ takes to output 10,000 different paths. This shows a difference of several orders of magnitude in runtime. The performance of K^* in this domain deserves attention. First, it performs much worse than $BELA_0$ in all maps. In fact, K_0 was requested only to find $\kappa = 1,000$ paths, yet it always takes significantly longer than $BELA_0$ takes to compute $\kappa = 10,000$ paths, even if it expands around the same number of nodes as shown in Figure 19. This indicates a difference in runtime of several orders of magnitude. Secondly, as conjectured in the roadmap domain, K_0's performance improves as the branching factor is reduced. As the percentage of obstruction increases, the

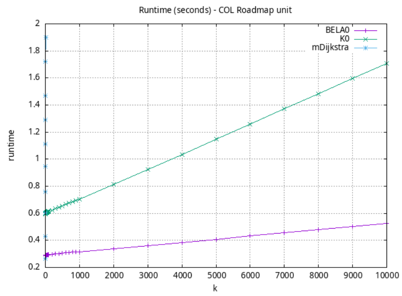
[4]https://movingai.com/benchmarks/grids.html



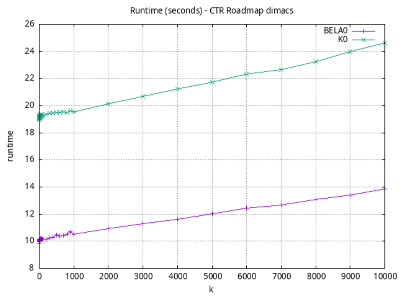
(a)



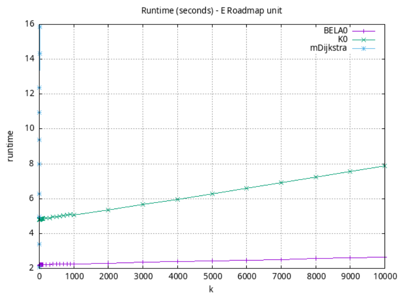
(b)



(c)



(d)



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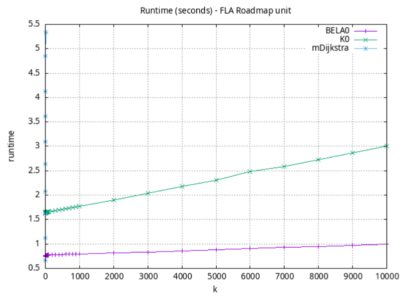

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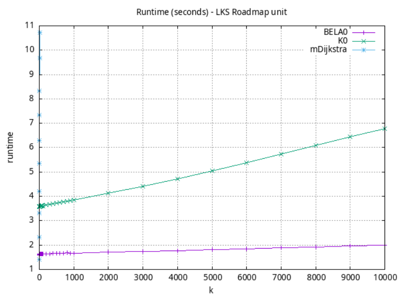
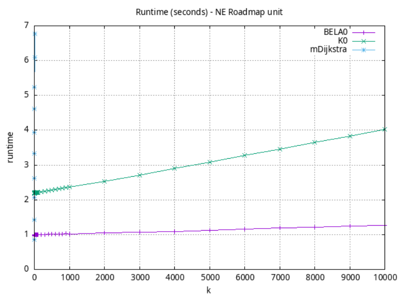
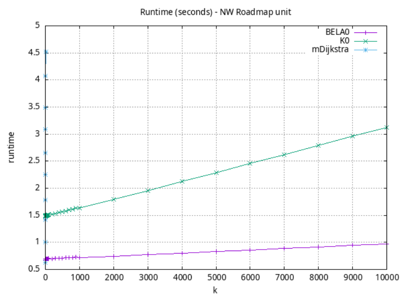


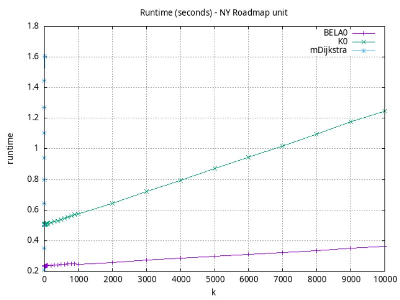
(g)



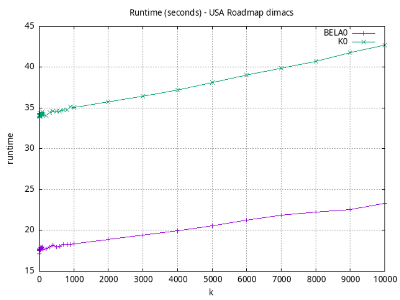
(h)



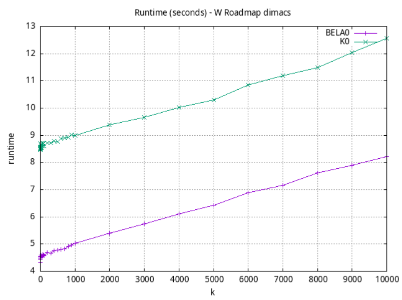
(i)



(j)



(k)



(l)

Figure 5: Runtime (in seconds) in the roadmap (dimacs) domain with brute-force search algorithms

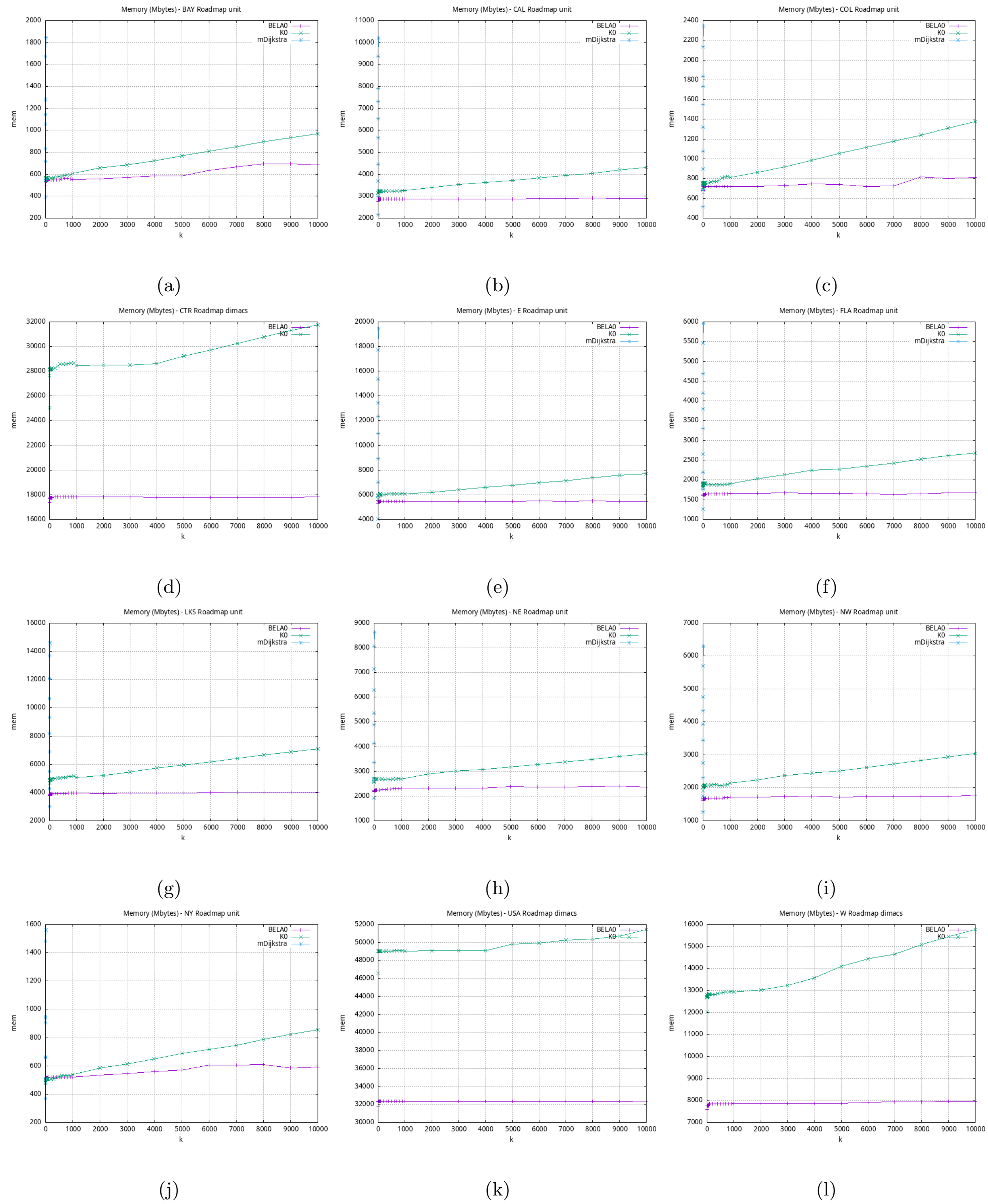
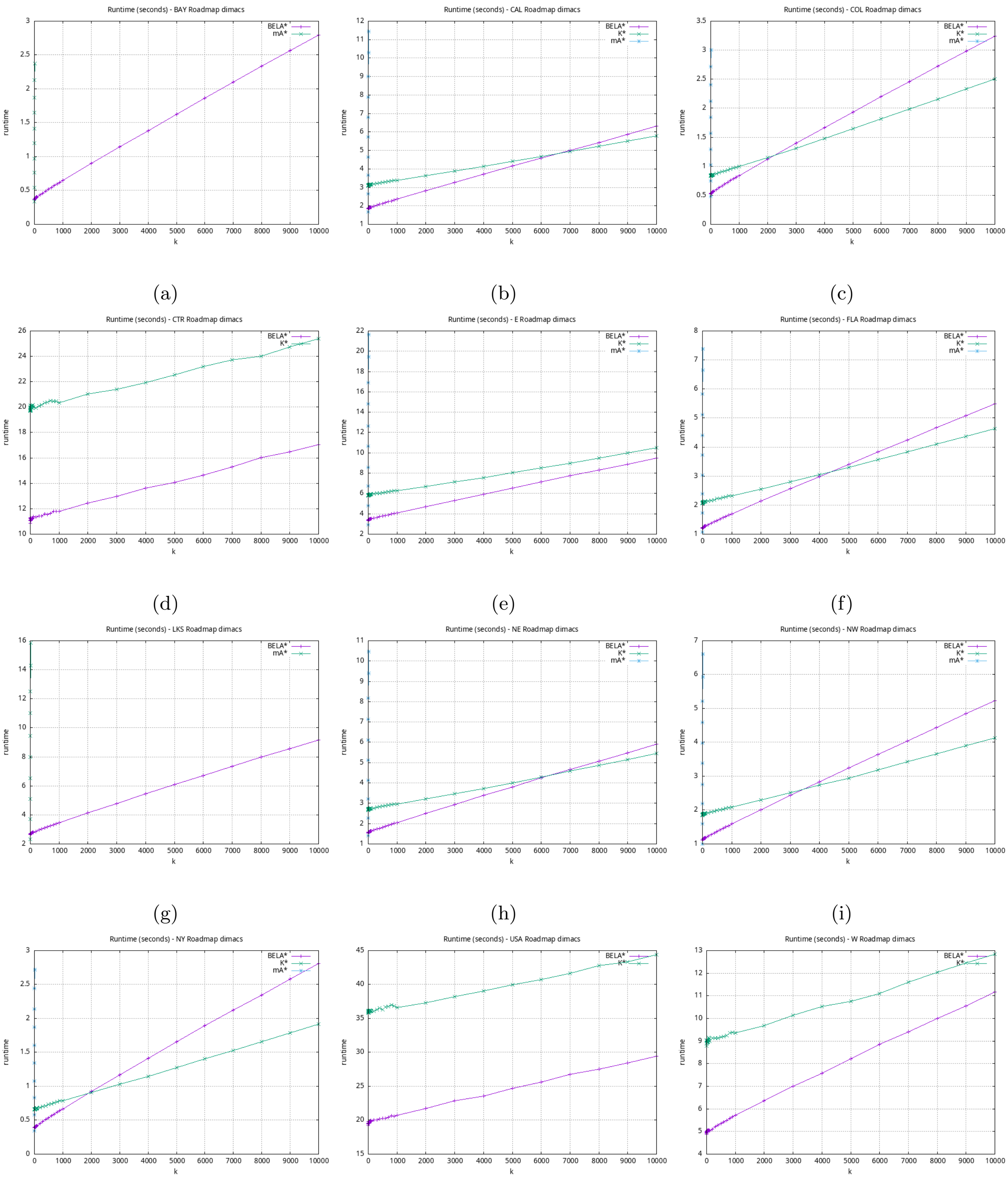


Figure 6: Memory usage (in Mbytes) in the roadmap (dimacs) domain with brute-force search algorithms

Figure 7: Number of expansions in the roadmap (dimacs) domain with brute-force search algorithms



(a) (b) (c)

(d) (e) (f)

(g) (h) (i)

(j) (k) (l)

Figure 8: Runtime (in seconds) in the roadmap (dimacs) domain with heuristic search algorithms

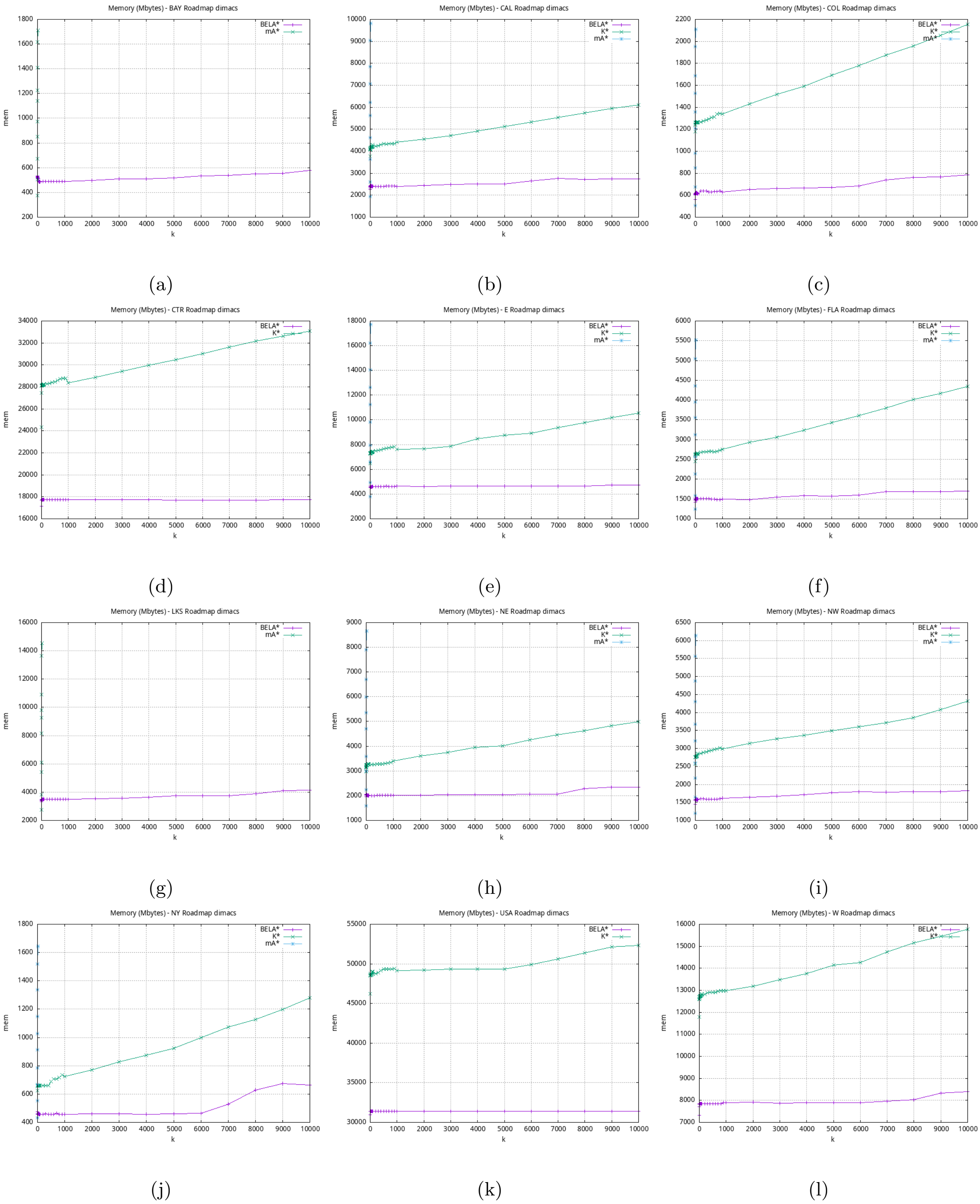
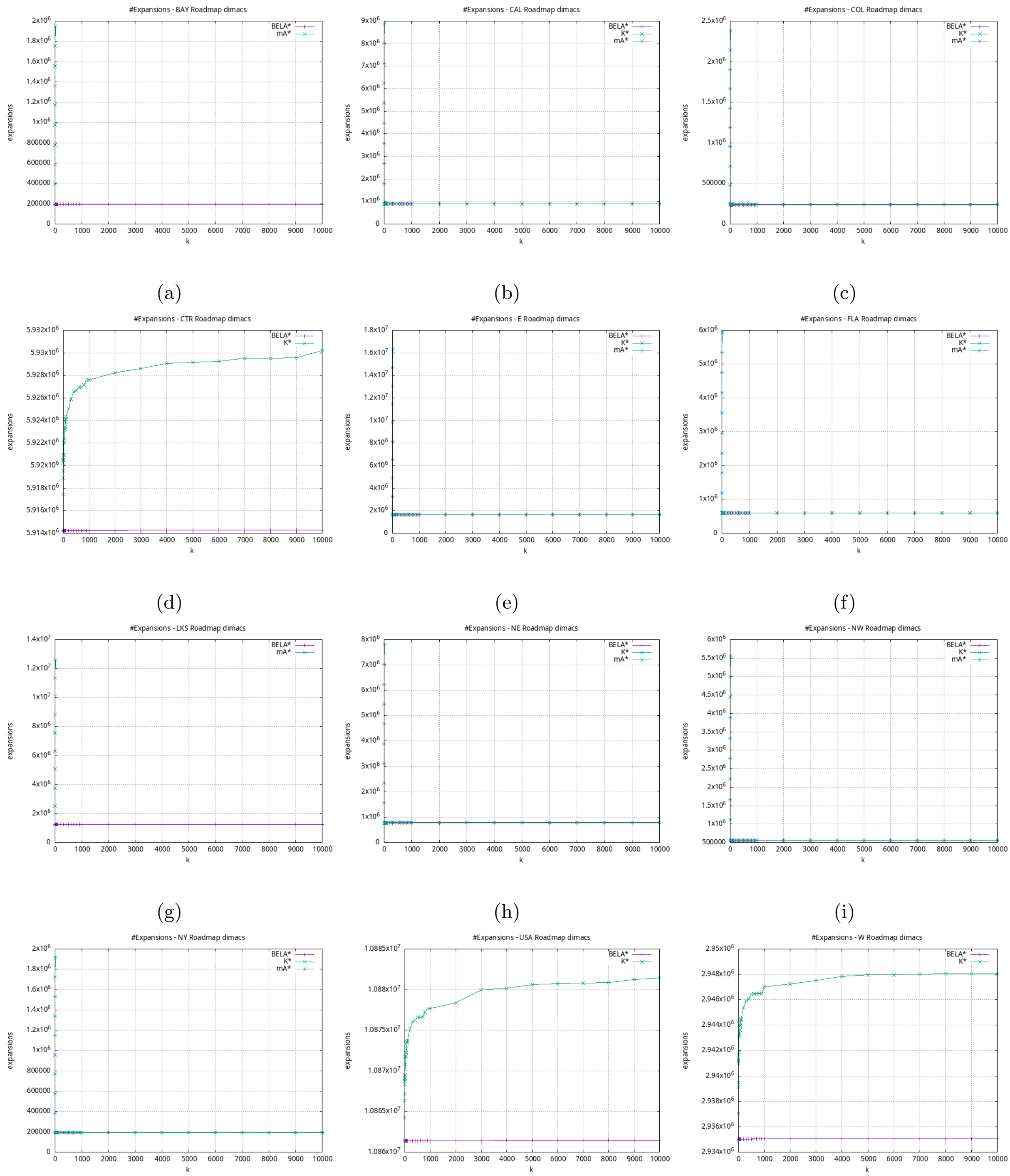


Figure 9: Memory usage (in Mbytes) in the roadmap (dimacs) domain with heuristic search algorithms



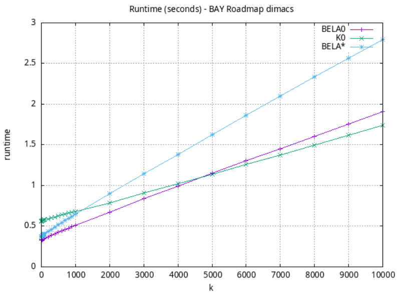
(a) (b) (c)

(d) (e) (f)

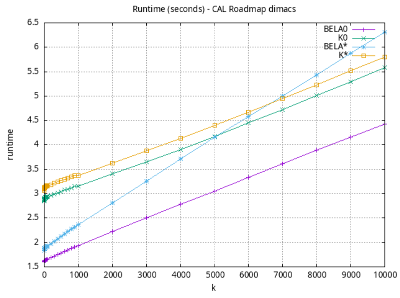
(g) (h) (i)

(j) (k) (l)

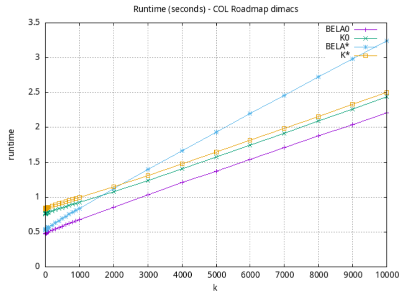
Figure 10: Number of expansions in the roadmap (dimacs) domain with heuristic search algorithms



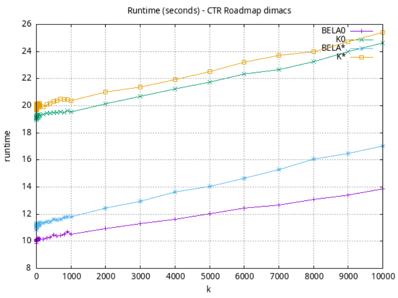
(a)



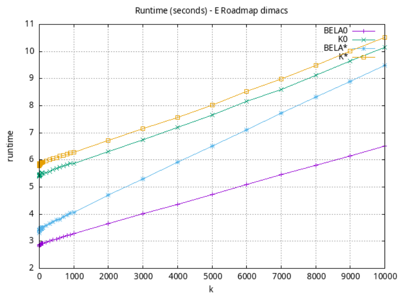
(b)



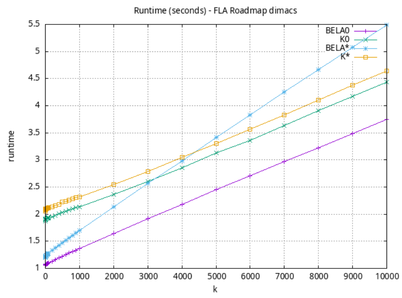
(c)



(d)



(e)



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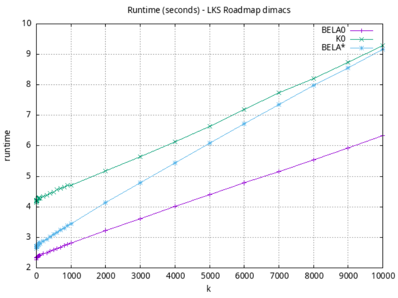
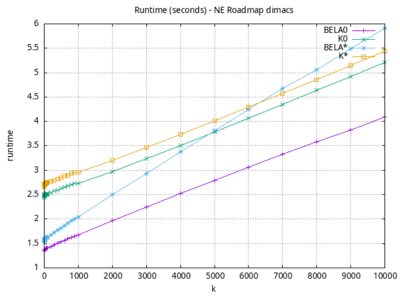
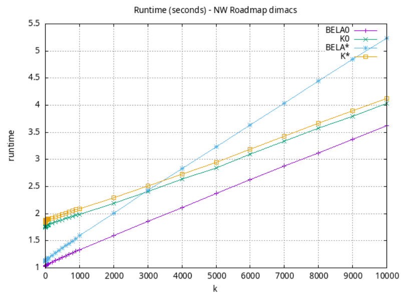


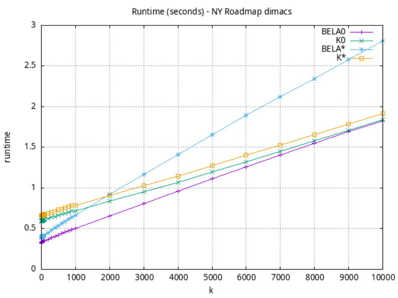
(g)



(h)



(i)



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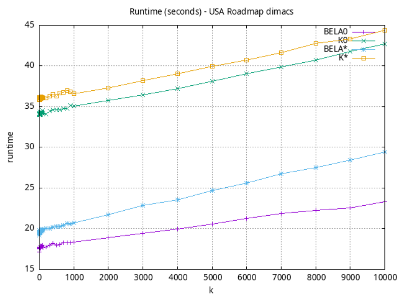
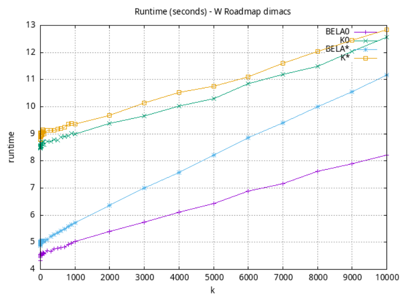


(k)



(l)

Figure 11: Runtime (in seconds) in the roadmap (dimacs) domain with mixed search algorithms

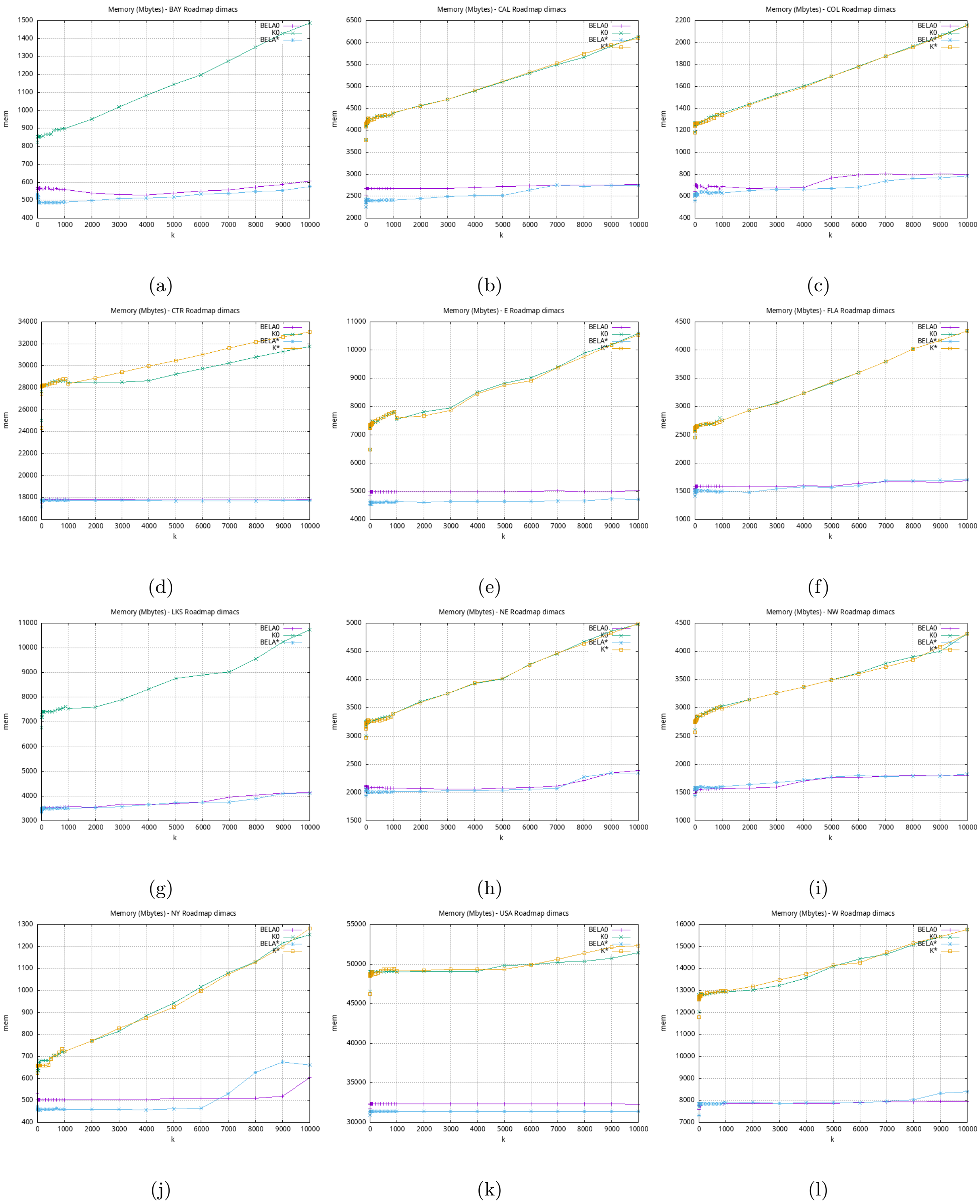
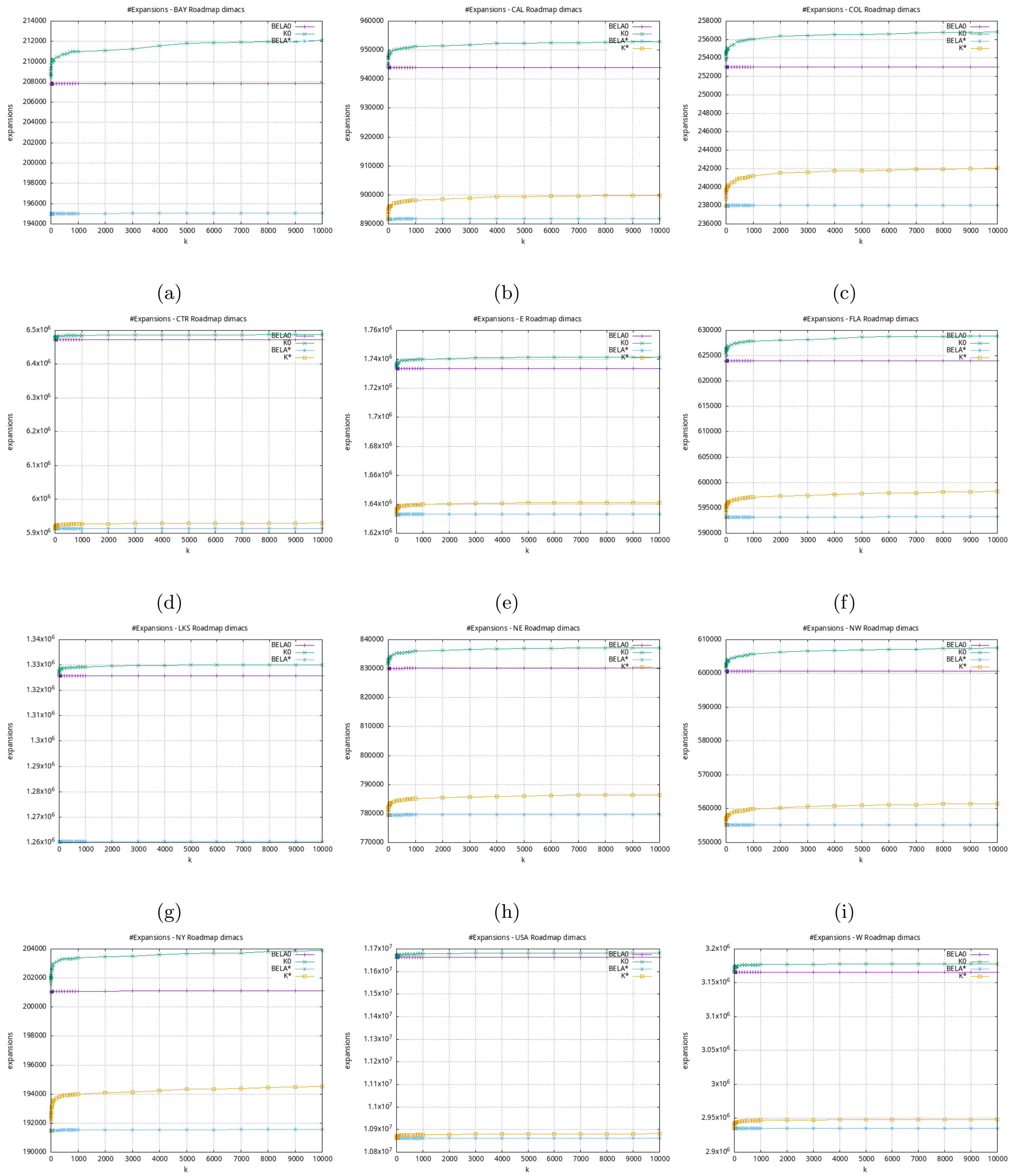


Figure 12: Memory usage (in Mbytes) in the roadmap (dimacs) domain with mixed search algorithms



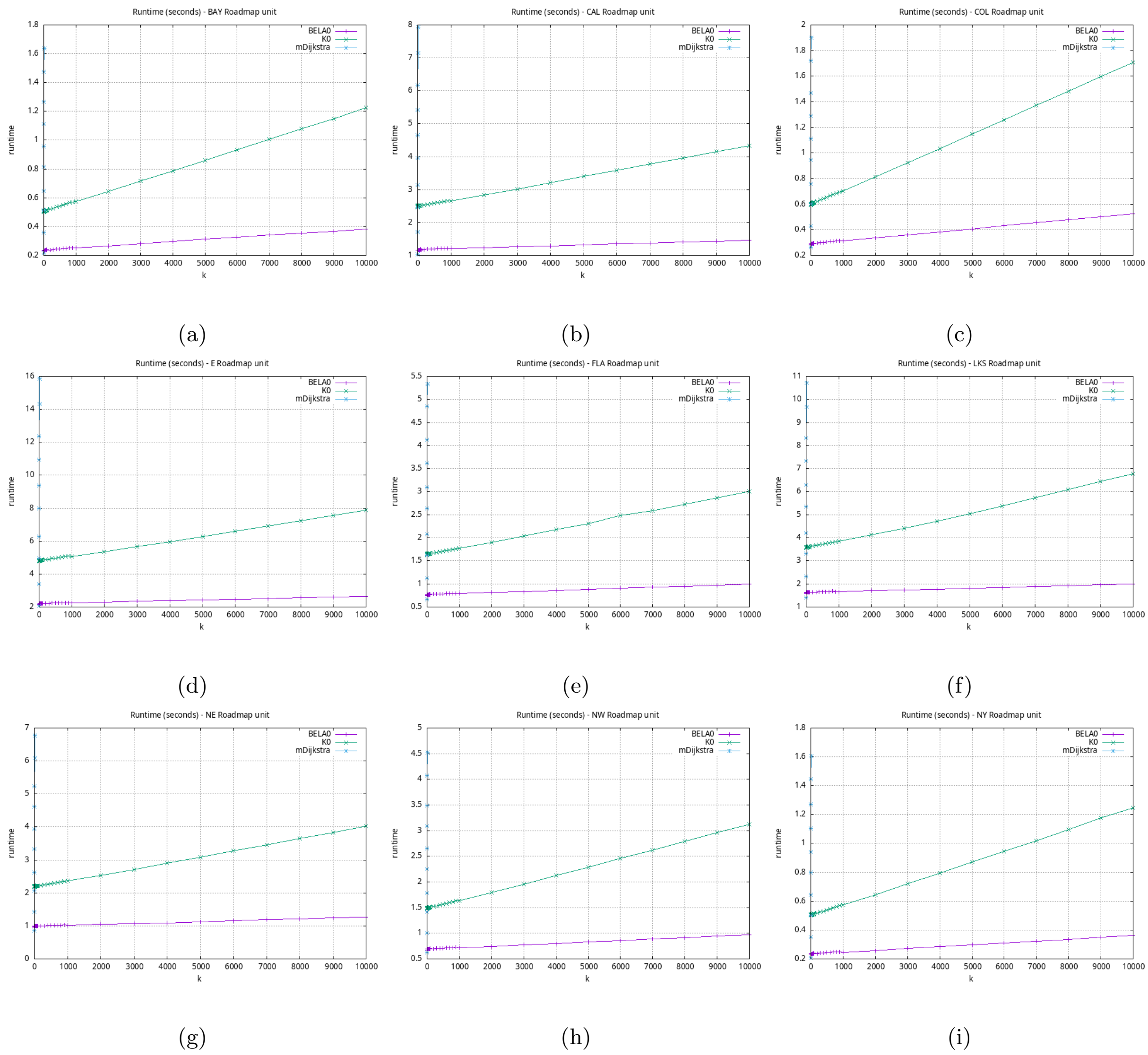
(a) (b) (c)

(d) (e) (f)

(g) (h) (i)

(j) (k) (l)

Figure 13: Number of expansions in the roadmap (dimacs) domain with mixed search algorithms

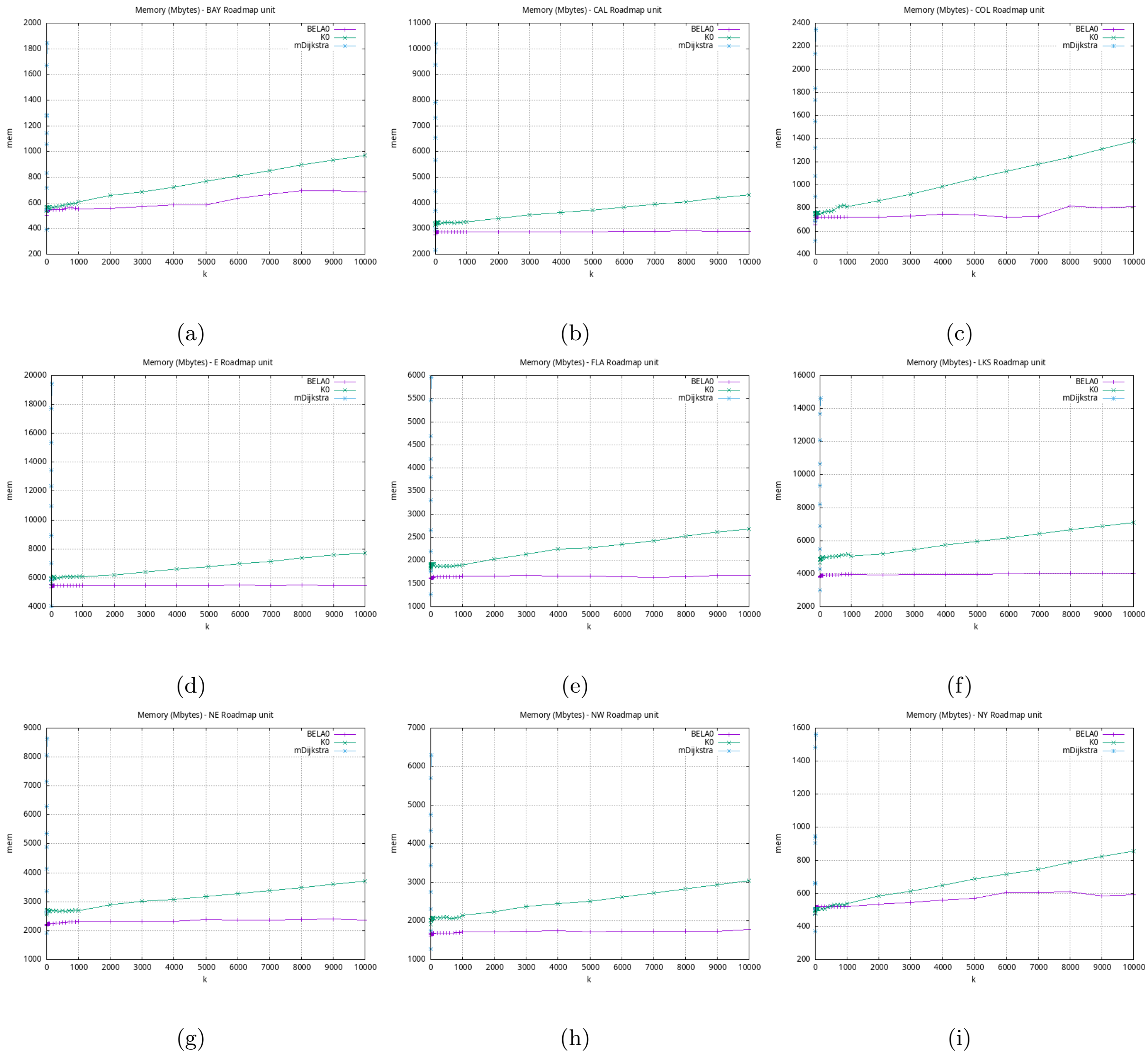


(a) (b) (c)

(d) (e) (f)

(g) (h) (i)

Figure 14: Runtime (in seconds) in the roadmap (unit) domain with brute-force search algorithms

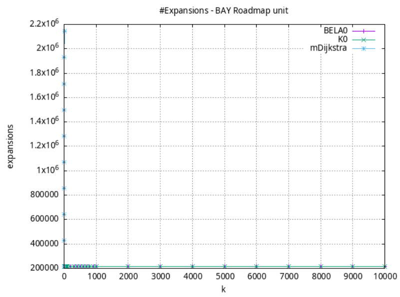


(a) (b) (c)

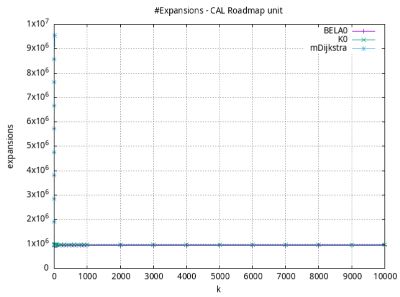
(d) (e) (f)

(g) (h) (i)

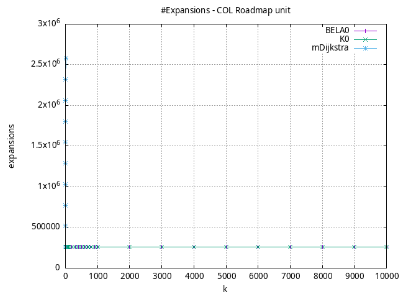
Figure 15: Memory usage (in Mbytes) in the roadmap (unit) domain with brute-force search algorithms



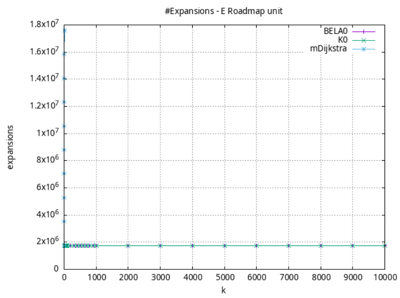
(a)



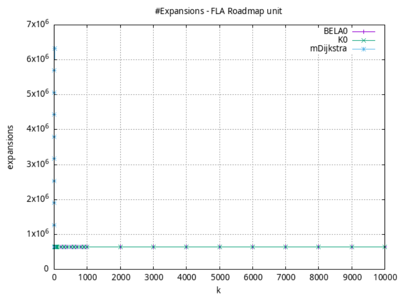
(b)



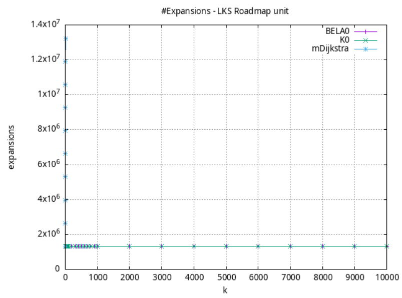
(c)



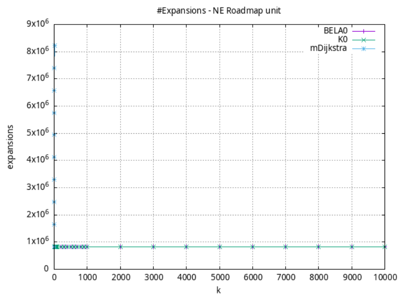
(d)



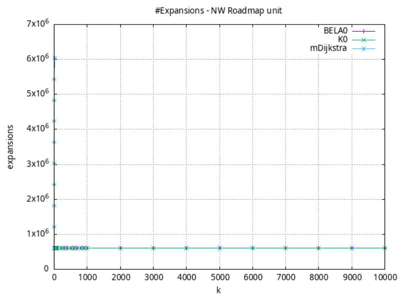
(e)



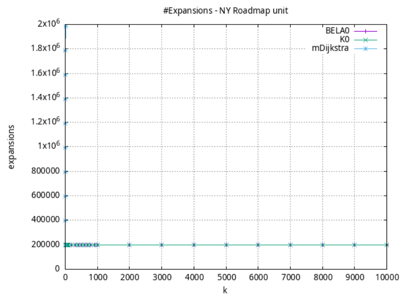
(f)



(g)

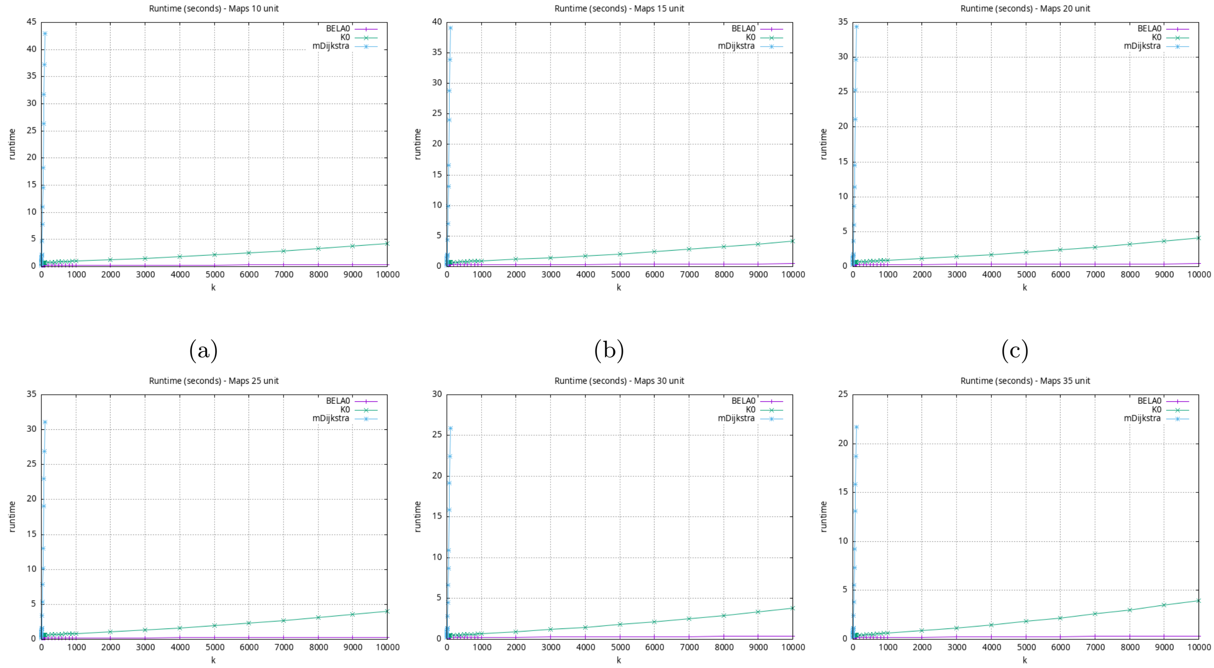


(h)



(i)

Figure 16: Number of expansions in the roadmap (unit) domain with brute-force search algorithms



(a) (b) (c)

(d) (e) (f)

Figure 17: Runtime (in seconds) in the maps (unit) domain with brute-force search algorithms

runtime improves. For example, it takes roughly 1 second to compute only 1,000 paths when 10% of the locations are occupied, but it can find the same number of paths in less than 0.65 seconds when the obstruction percentage gets to its maximum, 35.

When considering the application of the heuristic search algorithms, the differences between K_0 and $BELA_0$ become more acute, with a difference in runtime of one order of magnitude. K^* computes $\kappa = 1,000$ different paths in around the same amount of time it takes for $BELA^*$ to find ten times that amount of paths. Even if the heuristic funtion is not very well informed (in particular, for percentages of obstruction equal to 25 or larger), the difference in the number of expansions, shown in Figure 22 is of various orders of magnitude often, in particular in those cases with low percentages of obstruction. This difference is explained with an increase of the branching factor, which is conjectured to harm performance of K^* but, more importantly, by the observation that, in this domain in particular, the number of paths grow exponentially, so that a single centroid is enough to deliver even several billions of paths. $BELA^*$ can take advantage of this possibility and, in the end, it runs various orders of magnitude faster than K^*.

As for mA^*, the consideration of the heuristic function makes it improve its runtime marginally and it can now find $\kappa = 10$ solution paths. The reason for this low number is, as explained above, because the heuristic function is not very well informed.

6.2.2 Octile variant

In this variant, in addition to horizontal and vertical moves, it is also possible to move to cells diagonally adjacent to the current cell, provided they are not marked as inaccessible. This doubles the branching factor from 4 to 8. In addition, the octile variant is a non-unit domain because the diagonal moves have a cost equal to 14, whereas horizontal and vertical moves have a cost equal to 10 units. The heuristic function used is the octile distance.

This variant is harder than the previous variant for all algorithms. Again, mDijkstra is only able to find $\kappa = 4$ different paths, usually taking longer than the other algorithms which find either two orders of magnitude or even four orders of magnitude more paths in the same allotted time, as shown in Figure 23. This time, K_0 is restricted to find only $\kappa = 100$ different paths (10 times less than in the unit domain) and it consistently takes one order of magnitude more time than $BELA_0$, which computes $\kappa = 10,000$ solution paths. Even if $BELA_0$ also takes longer than it does in the unit variant, it still performs much better than all the other algorithms, being able to compute up to 10,000 paths in less than a second (averaged over each map). The difference in runtime between K_0 and $BELA_0$ can not be attributed neither to an increase in graph size (since they are the same than in the unit variant), nor the number of expansions performed by each algorithm. shown in Figure 25, since the difference is rather small. The degradation in performance of K_0 is therefore attributed to the increase in the branching factor which forces K_0 to consume more time in building and maintaining the path graph. Regarding $BELA_0$, its performance does not decrease significantly and, again, it delivers $\kappa = 10,000$ solution paths in less than a second on average across

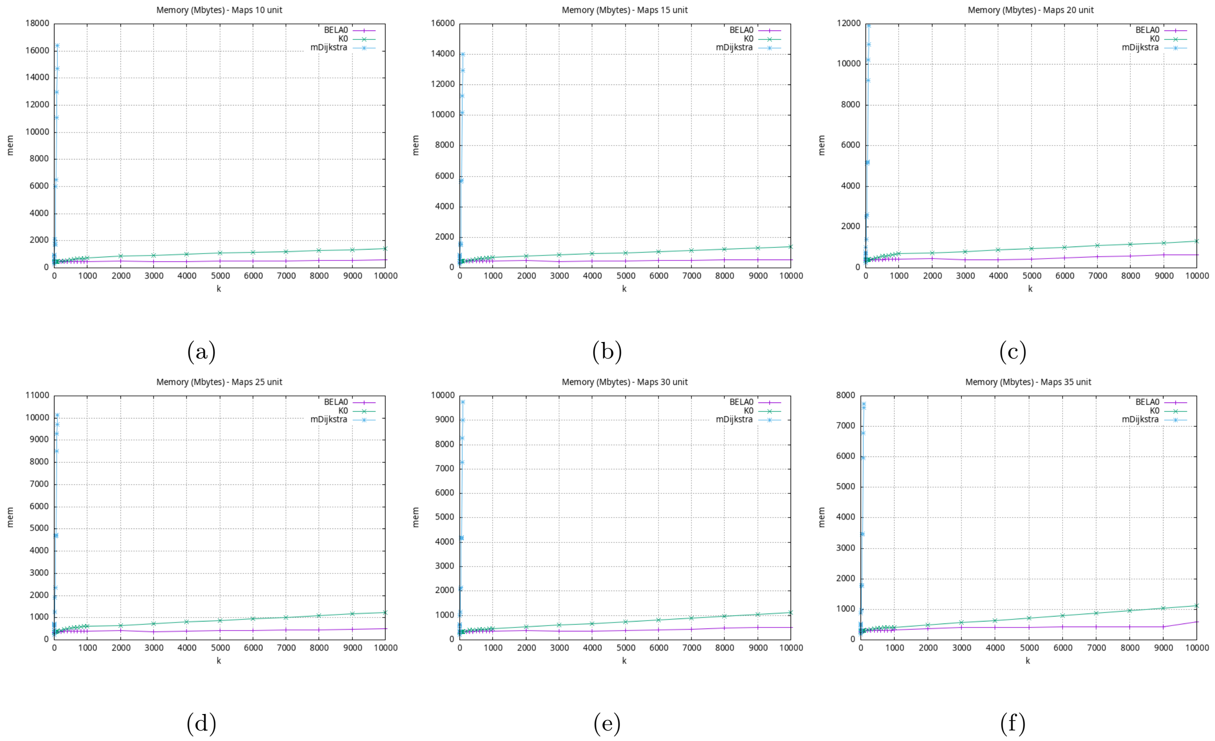


Figure 18: Memory usage (in Mbytes) in the maps (unit) domain with brute-force search algorithms

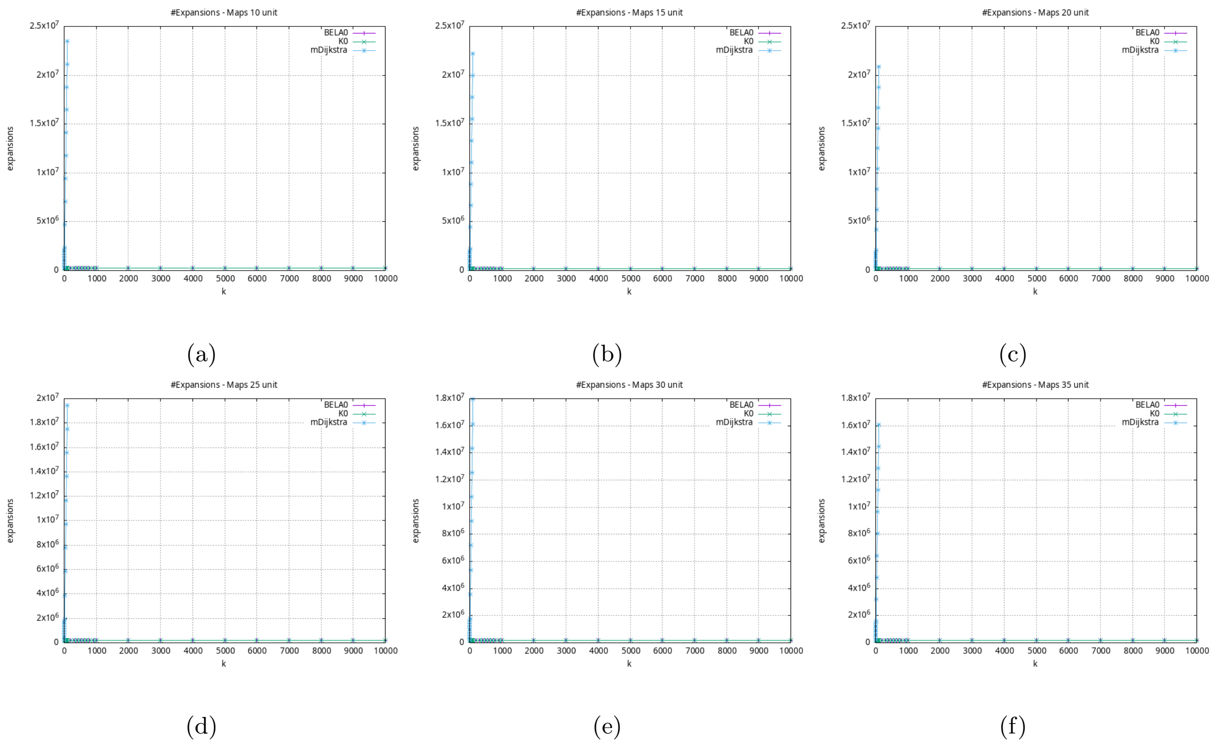


Figure 19: Number of expansions in the maps (unit) domain with brute-force search algorithms

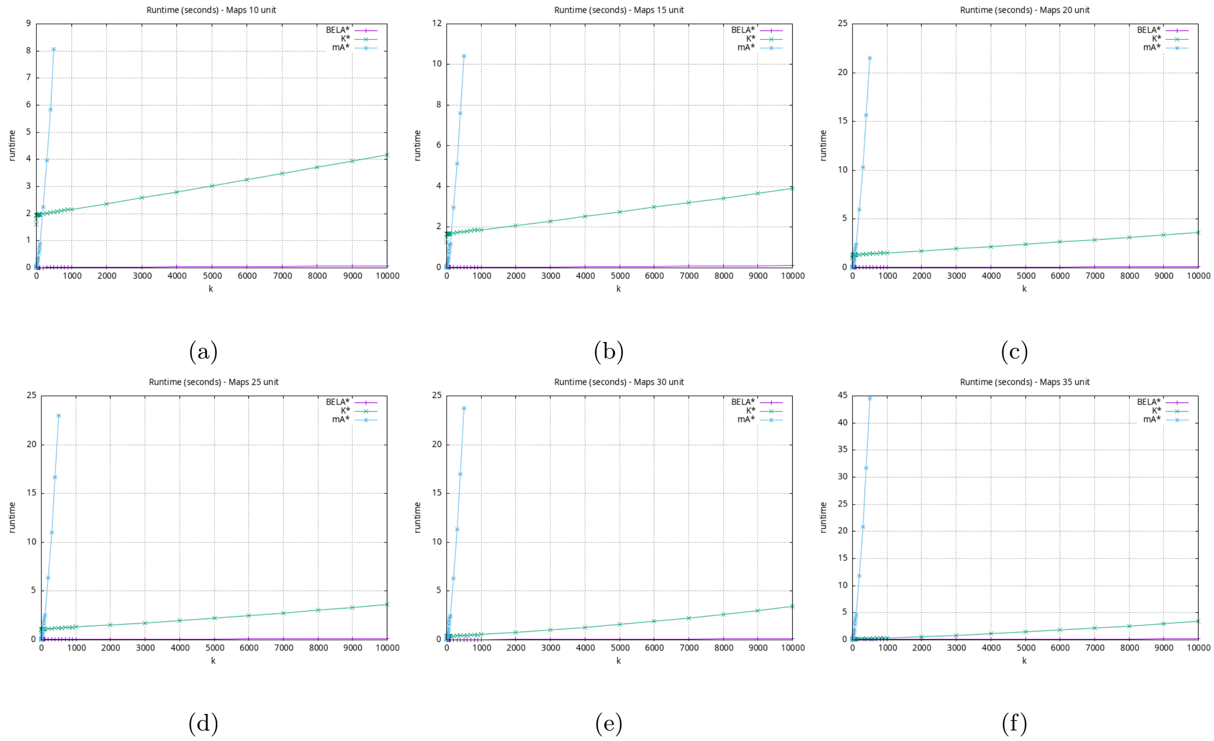


Figure 20: Runtime (in seconds) in the maps (unit) domain with heuristic search algorithms

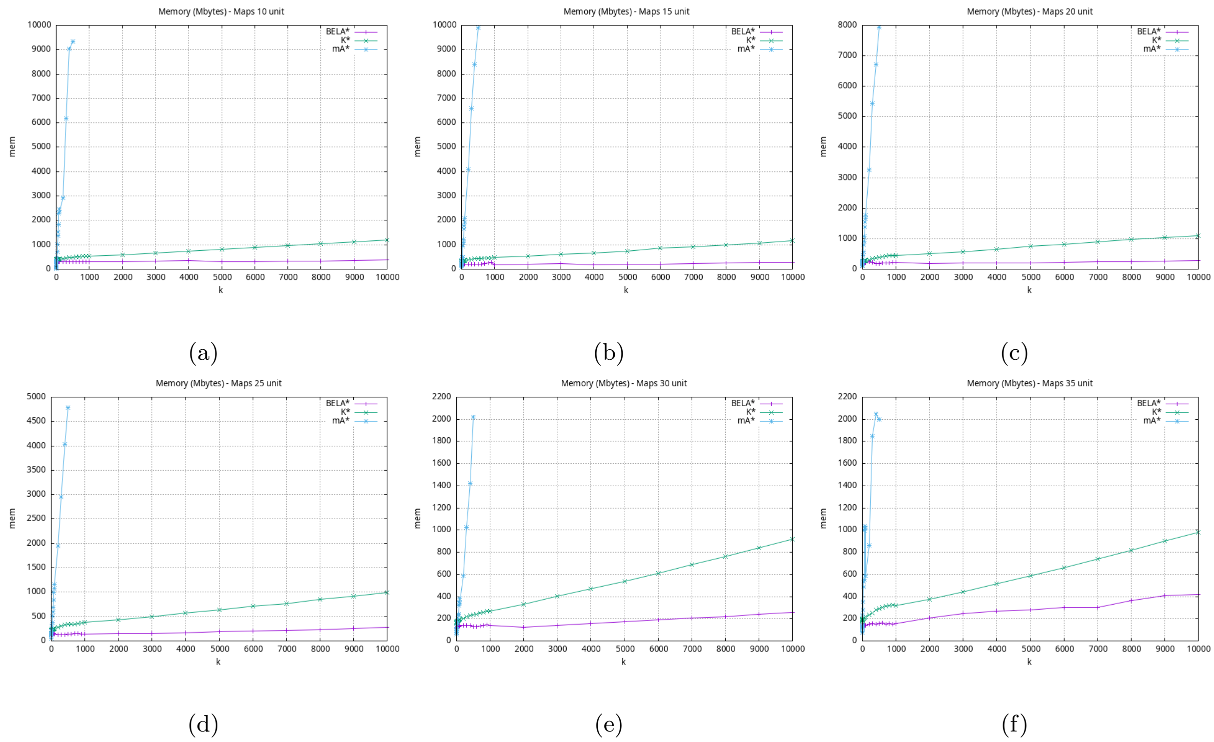
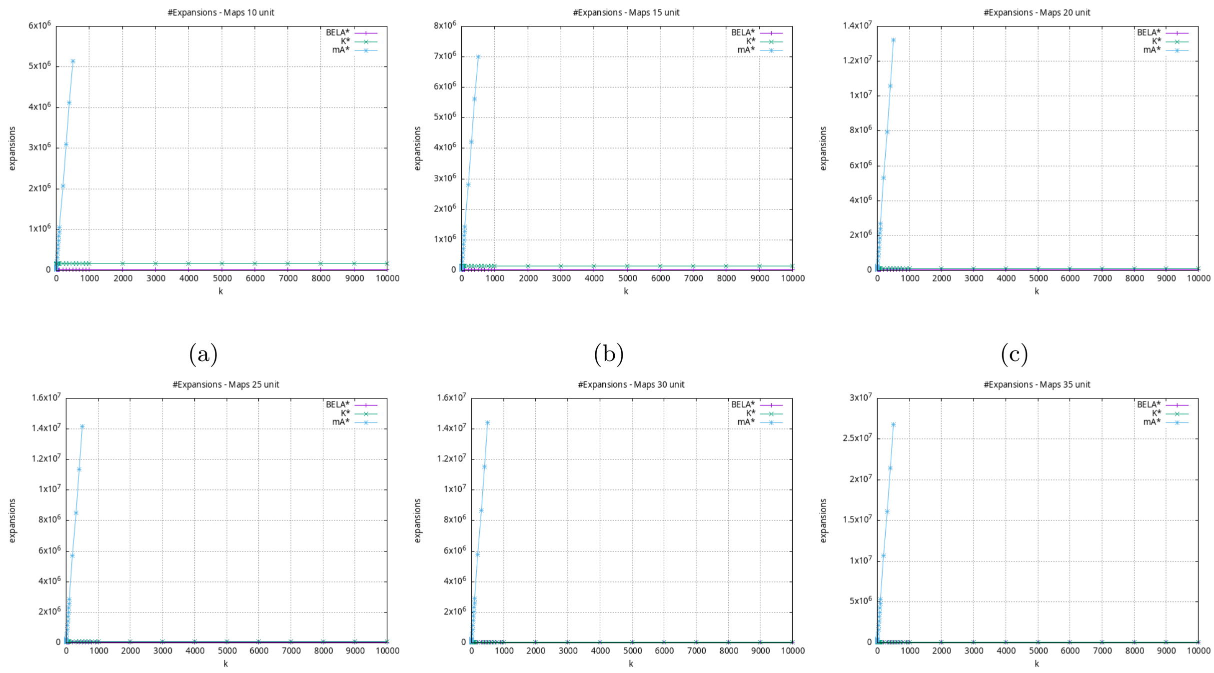


Figure 21: Memory usage (in Mbytes) in the maps (unit) domain with heuristic search algorithms



(a) (b) (c)

(d) (e) (f)

Figure 22: Number of expansions in the maps (unit) domain with heuristic search algorithms

all maps. One of the reasons for this performance is that in this variant, one single centroid suffices to deliver all solution paths.

6.3 N-Pancake

The N-Pancake domain defines a permutation state space with a size equal to $N!$. It is thus a significant challenge, as it also has a large branching-factor, $N-1$. The heuristic used is the GAP heuristic (Helmert, 2010), which is known to be very well informed in the unit variant discussed next. This allows current state-of-the-art solvers to solve instances of the 60-Pancake in less than 30 seconds on average per instance. In all cases, 100 instances were randomly generated and only those instances where the heuristic distance between the start state and the goal state was greater than or equal to $(N-2)$ were accepted. All of the points in the following plots have been averaged over 100 runs each. This domain has never been used, to the best of the authors' knowledge, as a testbed for algorithms solving the κ shortest path problem.

6.3.1 Unit variant

The unit variant is the classic version of the N-Pancake problem (Dweighter, 1975), where an arbitrary permutation of the symbols $\{1,\ldots,N\}$ has to be transformed into the identity permutation by performing prefix reversals, all of which have cost 1.

The brute-force variants of the algorithms under consideration were only tested on the 10-Pancake, because this state space is big enough for them, with 3,628,800 different states. The results are shown in Figures 29–31. Only BELA$_0$ was able to find 10 different paths in less than 25 seconds on average. In this domain, mDijkstra performed significantly better than K$_0$, but only for very low values of κ. In fact, mDijkstra was requested to find only $\kappa=3$ different solutions, because finding a fourth solution exhausts the available memory for some instances. K$_0$ performed much worse than BELA$_0$, doubling the number of expansions, being almost five times slower in the end, and taking also five times more memory than it.

Experiments using the GAP heuristic were particularly interesting. K* is indeed the worst algorithm in this domain. For example, in the 20-Pancake (see Figure 32a) it takes a huge amount of time for finding only $\kappa=10$ paths, while mA* and BELA* can find up to 1,000 solutions in much less time. Indeed, both mA* and BELA* are already two orders of magnitude faster with $\kappa=10$, the maximum value attempted with K*. In the end, BELA* is one order of mangitude faster for finding two orders of magnitude more solutions. This degradation in the running time of K* is attributed to two different factors: On one hand, the large branching factor which forces K* to spend much more time updating and maintaining its path graph; secondly, it expands significantly more nodes than BELA*, which is likely caused by the swapping criterion used. For the first time, mA* seems to be competitive with BELA*, even if it consistently performs worse than it over all values of κ. This behavior is due to the accuracy of the heuristic function. Observing the results in the 30 and 40-Pancake (see Figures 32b and 32c)

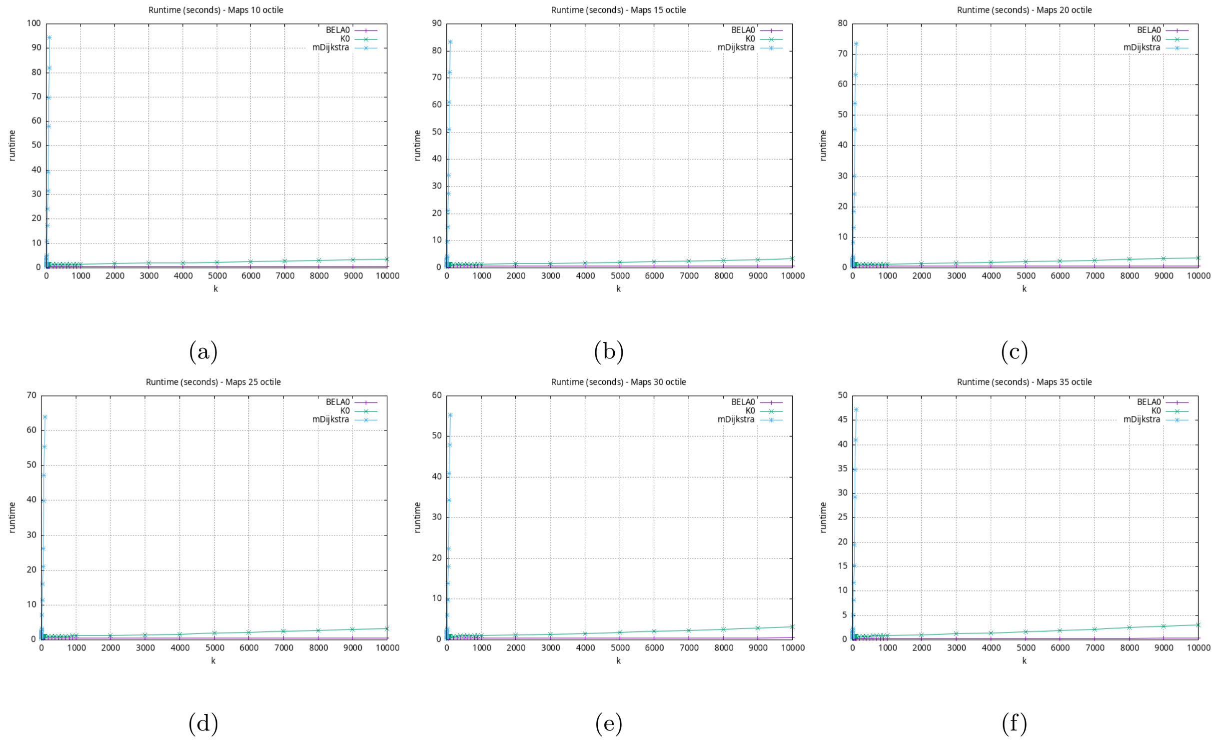


Figure 23: Runtime (in seconds) in the maps (octile) domain with brute-force search algorithms

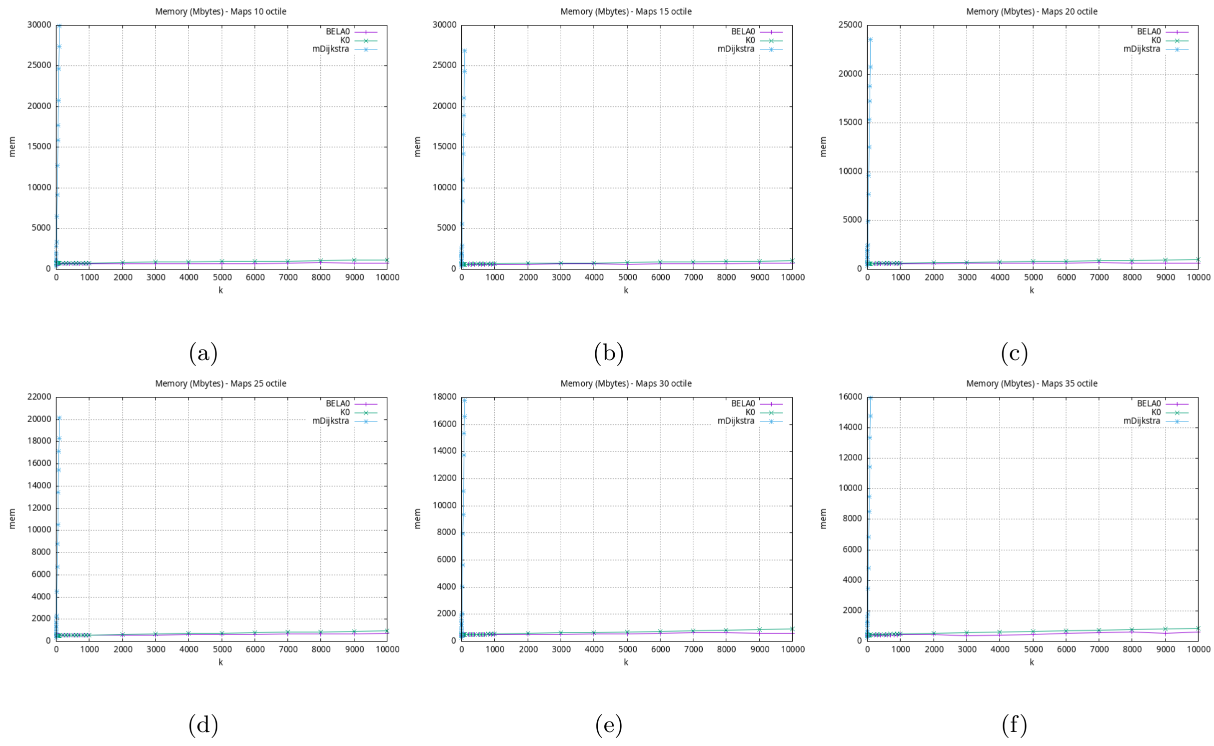
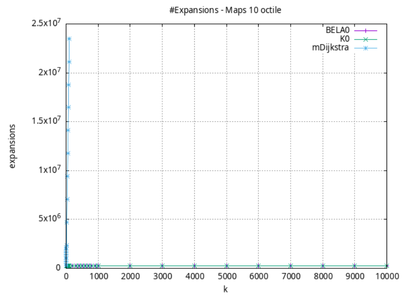
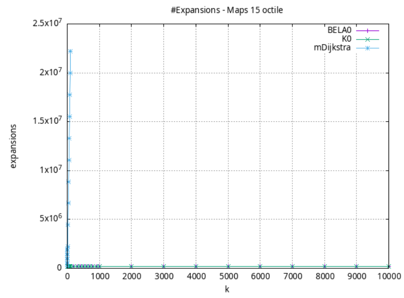


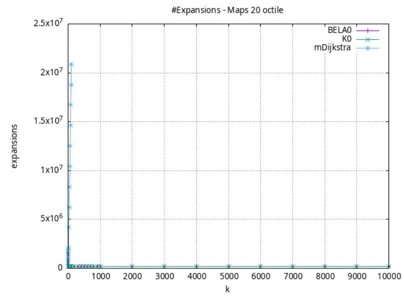
Figure 24: Memory usage (in Mbytes) in the maps (octile) domain with brute-force search algorithms



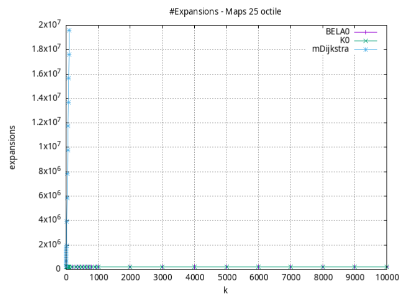
(a)



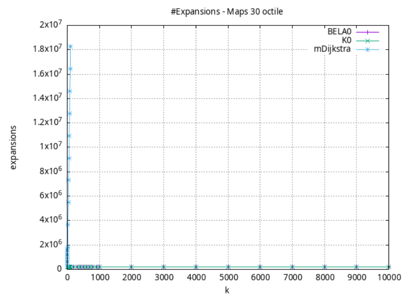
(b)



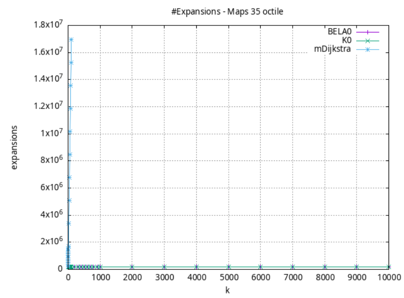
(c)



(d)

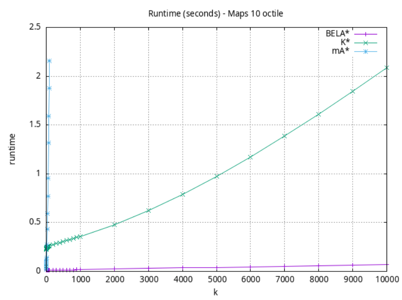


(e)

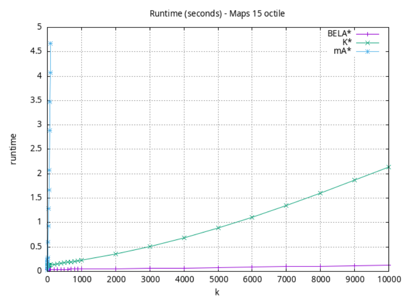


(f)

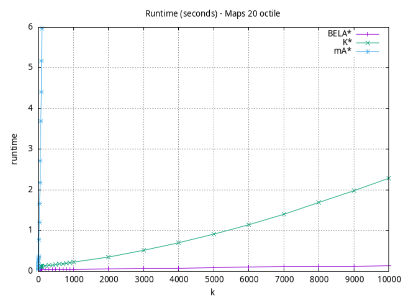
Figure 25: Number of expansions in the maps (octile) domain with brute-force search algorithms



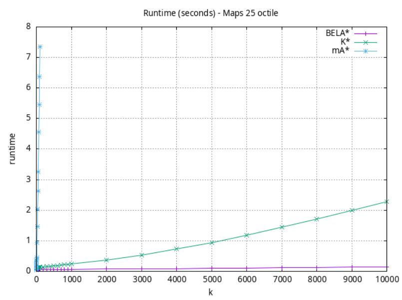
(a)



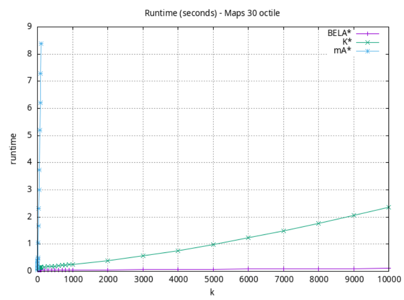
(b)



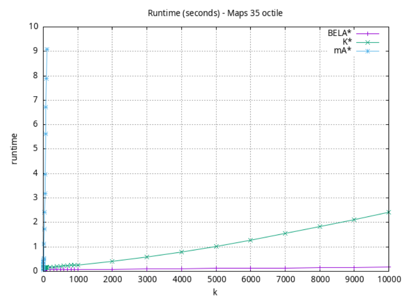
(c)



(d)



(e)



(f)

Figure 26: Runtime (in seconds) in the maps (octile) domain with heuristic search algorithms

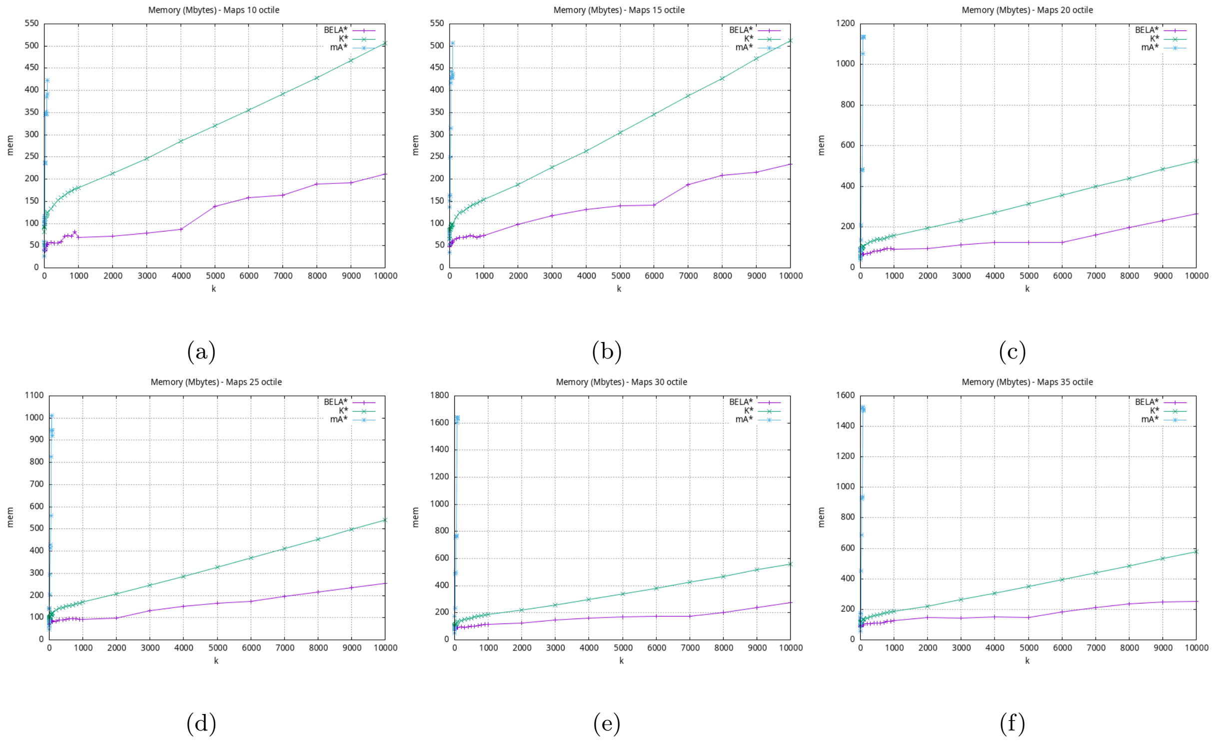


Figure 27: Memory usage (in Mbytes) in the maps (octile) domain with heuristic search algorithms

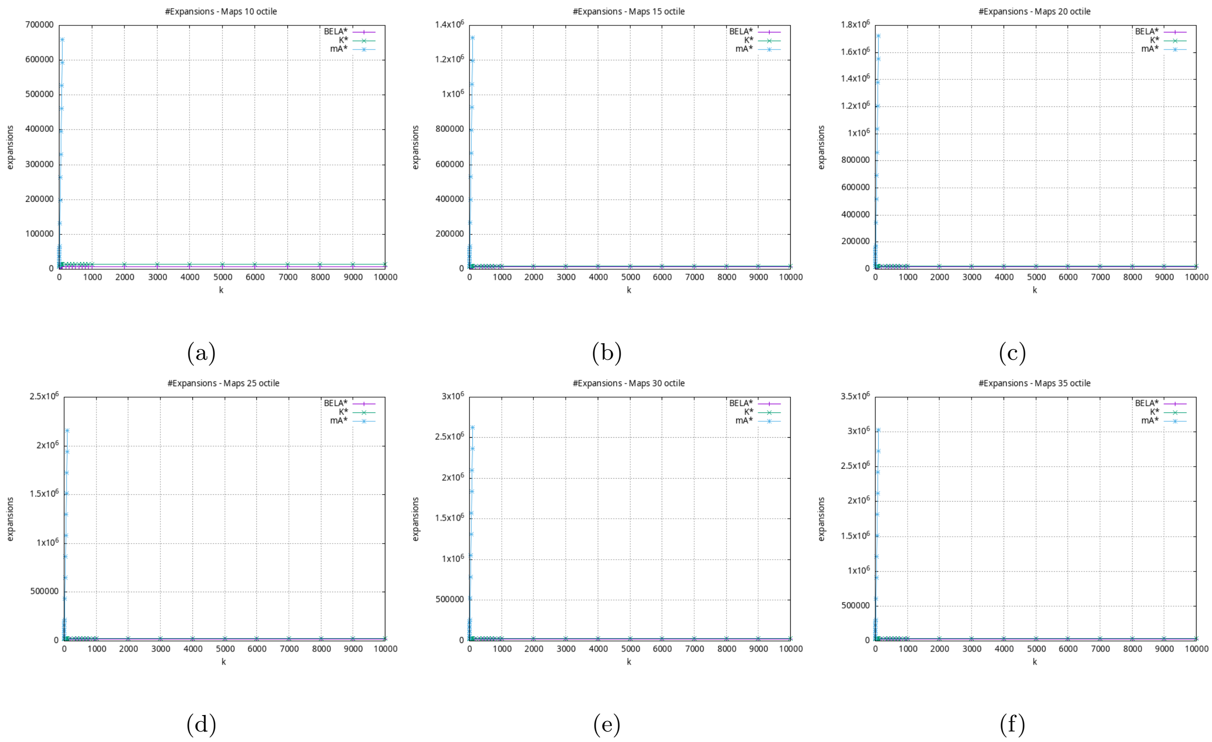
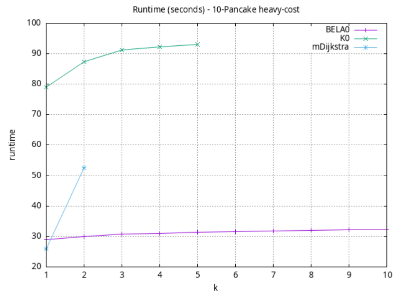
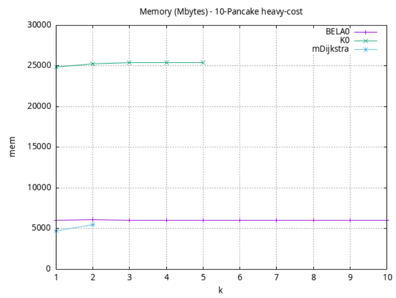


Figure 28: Number of expansions in the maps (octile) domain with heuristic search algorithms



(a)

Figure 29: Runtime (in seconds) in the n-pancake (unit) domain with brute-force search algorithms



(a)

Figure 30: Memory usage (in Mbytes) in the n-pancake (unit) domain with brute-force search algorithms

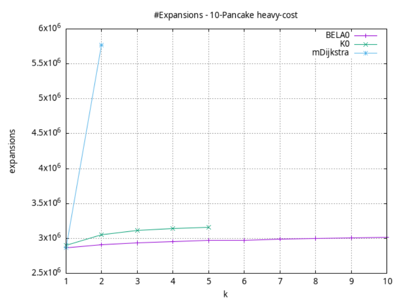
we can see that that the difference between mA*and BELA* increases with the growth of κ. mA* seems to be particularly competitive with BELA* in the 40-pancake where the latter is roughly 15% faster only. However, mA* takes more memory in this domain (see Figure 33c), and it exhausts all the available memory with $\kappa = 30$ while BELA* is able to find solutions up to $\kappa = 40$.

Note that in the 30 and 40-Pancake only values for $\kappa = 1$ are given for K*, which are one order of magnitude worse than the runtime of the other algorithms.

6.3.2 Heavy-cost variant

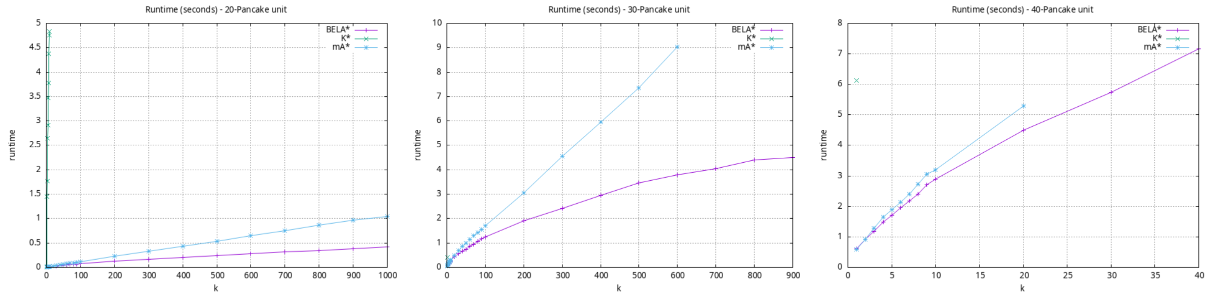
In the heavy-cost variant, the cost of each prefix reversal is defined as the size of the disc that becomes first in the permutation after the reversal. This variant is much harder than the unit version, because the GAP heuristic is not so well informed now, even if a weighted version of the GAP heuristic is being used. In the weighted variant of the GAP heuristic, each gap gets weighted by the size of the smaller disc adjacent to it. As a result of its hardness, experiments in the octile variant of the N-Pancake were conducted with 32Gb of RAM memory.

Figures 35–37 show the results using the brute-force search algorithms. As before, only the 10-Pancake was tested. As shown in Figure 35a BELA$_0$ takes an average time slightly above 30 seconds to find $\kappa = 10$ solutions, whereas mDijkstra can solve instances only with $\kappa \leq 2$ with a much worse average time than BELA$_0$ for $\kappa = 2$; K$_0$



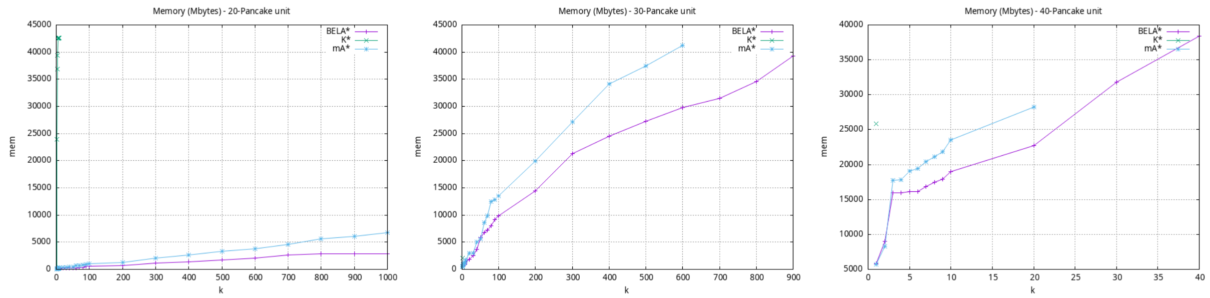
(a)

Figure 31: Number of expansions in the n-pancake (unit) domain with brute-force search algorithms



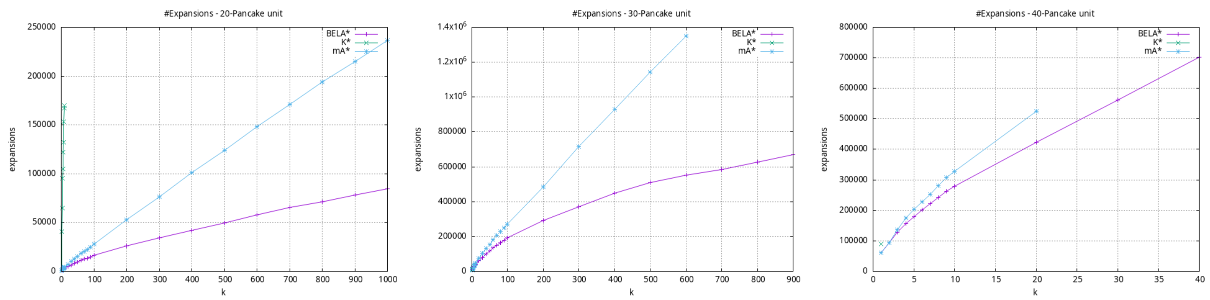
(a) (b) (c)

Figure 32: Runtime (in seconds) in the n-pancake (unit) domain with heuristic search algorithms



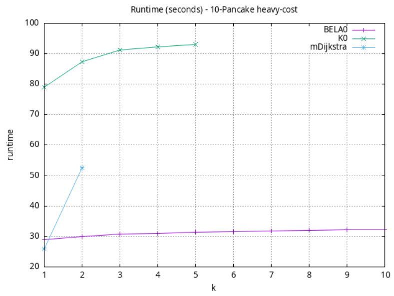
(a) (b) (c)

Figure 33: Memory usage (in Mbytes) in the n-pancake (unit) domain with heuristic search algorithms



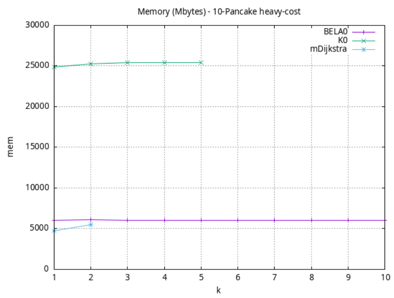
(a) (b) (c)

Figure 34: Number of expansions in the n-pancake (unit) domain with heuristic search algorithms



(a)

Figure 35: Runtime (in seconds) in the n-pancake (heavy-cost) domain with brute-force search algorithms

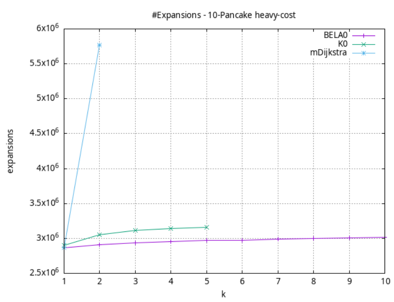


(a)

Figure 36: Memory usage (in Mbytes) in the n-pancake (heavy-cost) domain with brute-force search algorithms

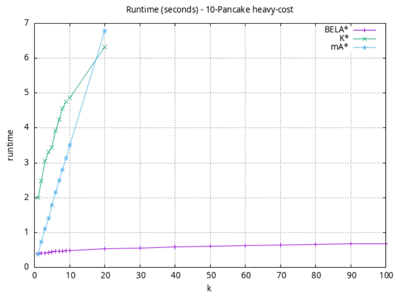
behaves also much worse than BELA_0 even if it manages to find solutions with up to $\kappa = 5$. Figure 35a clearly shows a trend where BELA_0 outperforms its contenders by a large margin in running time. As in the unit variant, K_0 expands significantly more nodes than BELA_0. It is important to remark that this figure indicates a general trend observed in most experiments throughout all domains, this is, that mDijkstra with $\kappa = 1$ is faster than BELA_0. This is not surprising at all, since mDijkstra with $\kappa = 1$ becomes vanilla Dijkstra with no significant overhead, whereas both BELA_0 and K_0 have an overhead necessary for efficiently solving problems with larger values of κ. Namely, the runtime cost originating from maintaining the closed lists in both algorithms. Nevertheless, just like most experiments conducted, mDijkstra (and also mA*) becomes immediately worse than BELA_0 (and BELA*, respectively) for low values of κ, even just 2, as shown in Figure 35a.

As a consequence of the degradation in performance of the heuristic function, experiments with the heavy-cost variant with the informed versions of all algorithms for values of N larger than 10 took too long and, in many cases memory was exhausted. For this reason, only experiments in the 10-Pancake were conducted, though with a larger value of κ, 100. Figures 38–40 show a dramatic difference in performance. Again, BELA* is one order of magnitude faster for finding up to one order of magnitude more solutions when compared to either mA* or K*: While BELA* finds 100 different solutions in less than a second on average, both mA* and K* take 3 and almost 5 seconds each on average respectively, for computing only 10 solutions.



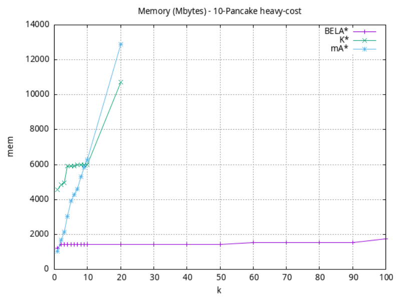
(a)

Figure 37: Number of expansions in the n-pancake (heavy-cost) domain with brute-force search algorithms



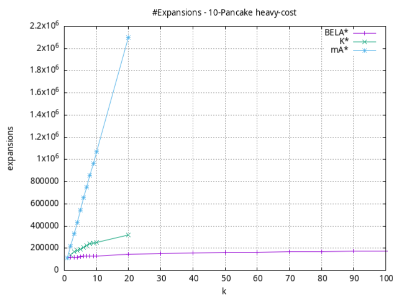
(a)

Figure 38: Runtime (in seconds) in the n-pancake (heavy-cost) domain with heuristic search algorithms



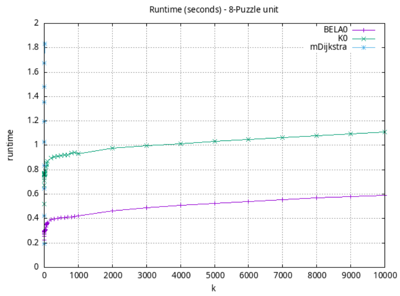
(a)

Figure 39: Memory usage (in Mbytes) in the n-pancake (heavy-cost) domain with heuristic search algorithms



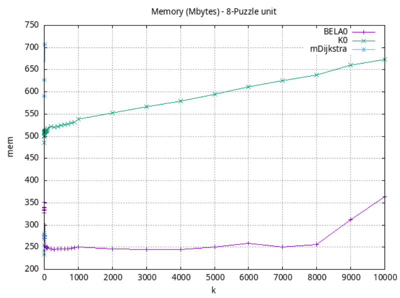
(a)

Figure 40: Number of expansions in the n-pancake (heavy-cost) domain with heuristic search algorithms



(a)

Figure 41: Runtime (in seconds) in the n-puzzle (unit) domain with brute-force search algorithms



(a)

Figure 42: Memory usage (in Mbytes) in the n-puzzle (unit) domain with brute-force search algorithms

6.4 N-Puzzle

The N-Puzzle is a classical combinatorial task (W. A. Johnson, 1879) that has a state space with $\frac{N^2!}{2}$ different states. Up to $N^2 - 1$ different symbols are arranged over a square matrix (though other arrangements are possible), leaving only one blank position, so that only symbols horizontally or vertically adjacent to it can swap their locations. The goal is to re-arrange all symbols into the identity permutation where the blank tile must be located in the upper-left corner. The 8-Puzzle and the 15-Puzzle were used for our experiments. In the first case, 100 random instances were randomly generated, whereas in the 15-Puzzle the 40 easiest instances of the Korf's test suite were selected (E. Korf, 1985). As a matter of fact, this test suite is known to extremely difficult for best-first search strategies when using the Manhattan distance (Burns et al., 2012), even without trying to find $\kappa > 1$ solution paths. As in the case of the N-Pancake, this is the first time this domain is used as a testbed for κ shortest path algorithms to the best of the authors' knowledge.

6.4.1 Unit variant

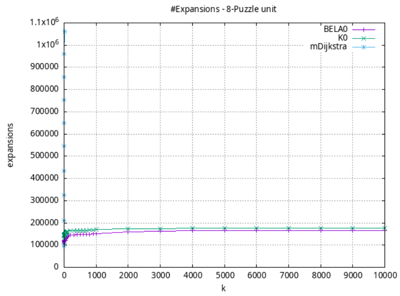
In the unit variant, all operators cost the same and thus, they are all equal to one. There are various heuristic functions for this domain. The current state-of-the-art uses Additive Pattern Databases (Felner et al., 2004). However, they are known to be inconsistent and thus they have been discarded for our experimentation, and the Manhattan distance is used instead.

Experiments with the brute-force variants were restricted to the 8-Puzzle, with 181,440 states. Figures 41-43 show the running time, memory usage and number of expansions. In this domain, K_0 is roughly twice as slow as BELA_0 for $\kappa = 10,000$, while mDijkstra performs very poorly due to the lack of a heuristic function.

Figures 44–46 show the results when using heuristic search algorithms, both in the 8-Puzzle (with $\kappa = 10,000$) and the 15-Puzzle —with $\kappa = 100$. The results in the 15-Puzzle (see Figure 44b) show huge improvements in running time when using BELA*, which finds the best 100 solutions in roughly 5 seconds on average, whereas both K* and mA* take one order of magnitude more time for very low values of κ. As observed in Figures 45 and 46, the profiles shown in running time are closely followed by those for memory usage and the number of expansions.

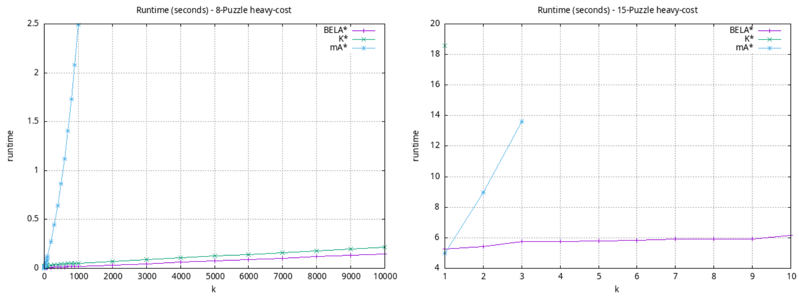
6.4.2 Heavy-cost variant

In the heavy-cost variant, the cost of a movement is equal to the content of the tile exchanged with the blank. A weighted variant of the Manhattan distance, where the distance of each tile is multiplied by its content is used as our heuristic. The resulting variant is much harder than the previous one, and thus only experiments with the



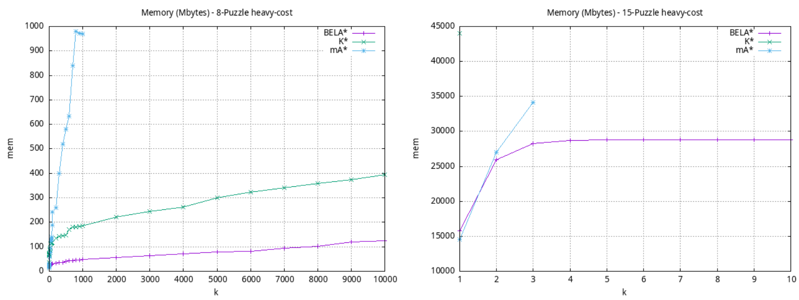
(a)

Figure 43: Number of expansions in the n-puzzle (unit) domain with brute-force search algorithms



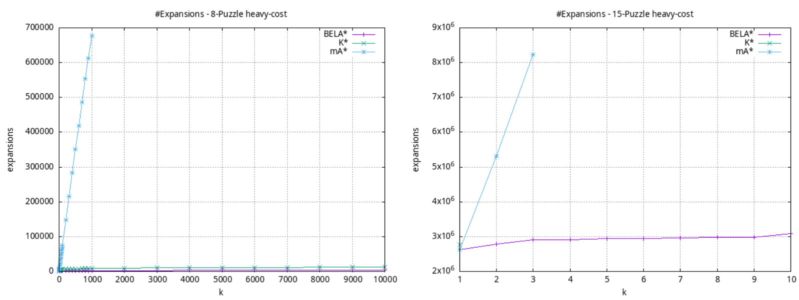
(a) (b)

Figure 44: Runtime (in seconds) in the n-puzzle (unit) domain with heuristic search algorithms



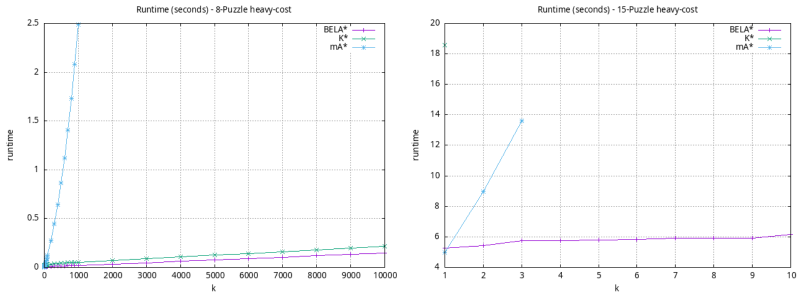
(a) (b)

Figure 45: Memory usage (in Mbytes) in the n-puzzle (unit) domain with heuristic search algorithms



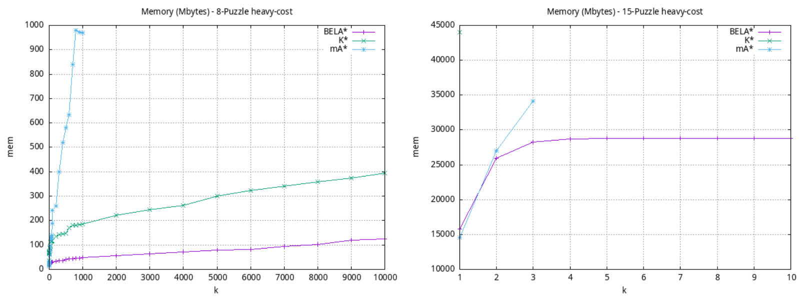
(a) (b)

Figure 46: Number of expansions in the n-puzzle (unit) domain with heuristic search algorithms



(a) (b)

Figure 47: Runtime (in seconds) in the n-puzzle (heavy-cost) domain with heuristic search algorithms



(a) (b)

Figure 48: Memory usage (in Mbytes) in the n-puzzle (heavy-cost) domain with heuristic search algorithms

heuristic versions were conducted.

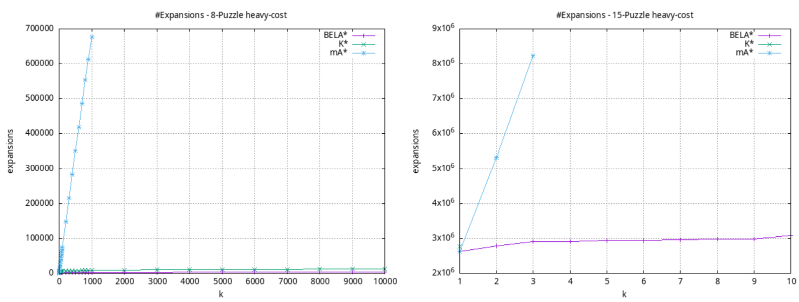
Results in the 8-Puzzle are almost identical to those in the unit variant — compare Figures 44 and 47. The reason is that the state space of the 8-Puzzle is too small as to pose any significant challenge when using a heuristic. Things change entirely when considering the 15-Puzzle, see Figure 47b: K* takes almost 20 seconds on average to find a single optimal solution, and mA* is able to compute the three best solutions in almost 14 seconds; BELA*, however, finds the best 10 solutions in roughly 6 seconds on average. The profiles shown in Figure 47b show differences of various orders of magnitude in running time.

A Tables

This appendix provides the same information given in section 6 but in tabular form for a selected collection of κ values. Only runtime is provided. In all cases, the best result is shown in boldface.

A.1 Roadmap

Tables 2–46 summarize the results in the roadmap domain for all variants and graphs. The size of each graph is given in Table 1.



(a) (b)

Figure 49: Number of expansions in the n-puzzle (heavy-cost) domain with heuristic search algorithms

Runtime (seconds) - BAY Roadmap unit								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
BELA$_0$	0.23	**0.23**	**0.24**	**0.24**	**0.25**	**0.25**	**0.31**	**0.38**
K$_0$	0.50	0.51	0.51	0.51	0.54	0.57	0.86	1.22
mDijkstra	**0.22**	1.64	–	–	–	–	–	–

Table 2: Runtime (in seconds) in the roadmap (dimacs) domain with brute-force search algorithms

Runtime (seconds) - CAL Roadmap unit								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
BELA$_0$	1.15	**1.16**	**1.17**	**1.17**	**1.20**	**1.21**	**1.32**	**1.46**
K$_0$	2.49	2.50	2.52	2.52	2.59	2.66	3.40	4.34
mDijkstra	**1.04**	7.92	–	–	–	–	–	–

Table 3: Runtime (in seconds) in the roadmap (dimacs) domain with brute-force search algorithms

A.1.1 9th DIMACS Challenge

The first tables provide all results in tabular form in the DIMACS domain: Tables 2–13 show the runtime of the brute-force variants; Tables 14–25 show the runtime of the heuristics variants, and Tables 26–37 show a comparison among both brute-force and heuristic variants.

A.1.2 Unit variant

Next, Tables 38–46 show the results in the unit variant of the roadmap domain.

A.2 Maps

Tables 47–70 show the runtime of all algorithms in the maps domain.

A.2.1 Unit variant

Tables 47–52 show the results of the brute-force variants being compared, whereas Tables 53–58 show the runtime for the heuristic variants, in the unit variant for all sizes of maps being tested.

A.2.2 Octile variant

Tables 59–64 show the runtime of the brute-force variants, and Tables 65–70 show the same statistics for the heuristic variants, in the octile variant for all sizes of maps being tested.

A.3 N-Pancake

Tables 71–76 show the runtime of all algorithms in the N-Pancake domain.

A.3.1 Unit variant

Table 71 shows the runtime of the brute-force search algorithms tested in the 10-Pancake in the unit variant. Tables 72–74 show the runtime of the heuristic search algorithms in the 20-, 30- and 40-Pancake, respectively, in the unit variant.

Runtime (seconds) - COL Roadmap unit								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
BELA$_0$	0.29	**0.29**	**0.29**	**0.29**	**0.30**	**0.31**	**0.41**	**0.53**
K$_0$	0.59	0.60	0.61	0.61	0.65	0.70	1.15	1.71
mDijkstra	**0.26**	1.90	–	–	–	–	–	–

Table 4: Runtime (in seconds) in the roadmap (dimacs) domain with brute-force search algorithms

Runtime (seconds) - CTR Roadmap dimacs								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	**9.85**	**10.04**	**10.17**	**10.17**	**10.48**	**10.52**	**12.01**	**13.85**
K_0	18.97	19.09	19.16	19.31	19.50	19.54	21.73	24.63

Table 5: Runtime (in seconds) in the roadmap (dimacs) domain with brute-force search algorithms

Runtime (seconds) - E Roadmap unit								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	2.15	**2.17**	**2.22**	**2.22**	**2.24**	**2.26**	**2.43**	**2.65**
K_0	4.79	4.81	4.84	4.84	4.96	5.07	6.28	7.87
mDijkstra	**2.05**	15.85	–	–	–	–	–	–

Table 6: Runtime (in seconds) in the roadmap (dimacs) domain with brute-force search algorithms

Runtime (seconds) - FLA Roadmap unit								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	0.74	**0.77**	**0.77**	**0.77**	**0.78**	**0.79**	**0.88**	**1.00**
K_0	1.63	1.64	1.66	1.66	1.71	1.77	2.31	3.00
mDijkstra	**0.67**	5.34	–	–	–	–	–	–

Table 7: Runtime (in seconds) in the roadmap (dimacs) domain with brute-force search algorithms

Runtime (seconds) - LKS Roadmap unit								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	1.58	**1.63**	**1.63**	**1.64**	**1.65**	**1.67**	**1.81**	**2.00**
K_0	3.56	3.58	3.60	3.61	3.72	3.84	5.05	6.78
mDijkstra	**1.41**	10.73	–	–	–	–	–	–

Table 8: Runtime (in seconds) in the roadmap (dimacs) domain with brute-force search algorithms

Runtime (seconds) - NE Roadmap unit								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	0.97	**0.99**	**1.00**	**1.00**	**1.01**	**1.02**	**1.13**	**1.27**
K_0	2.18	2.19	2.21	2.21	2.29	2.36	3.09	4.02
mDijkstra	**0.86**	6.76	–	–	–	–	–	–

Table 9: Runtime (in seconds) in the roadmap (dimacs) domain with brute-force search algorithms

Runtime (seconds) - NW Roadmap unit								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	0.68	**0.69**	**0.69**	**0.70**	**0.71**	**0.72**	**0.83**	**0.97**
K_0	1.48	1.49	1.50	1.51	1.56	1.64	2.29	3.13
mDijkstra	**0.62**	4.52	–	–	–	–	–	–

Table 10: Runtime (in seconds) in the roadmap (dimacs) domain with brute-force search algorithms

Runtime (seconds) - NY Roadmap unit								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	0.23	**0.23**	**0.24**	**0.23**	**0.24**	**0.25**	**0.30**	**0.36**
K_0	0.50	0.50	0.51	0.51	0.54	0.57	0.87	1.25
mDijkstra	**0.21**	1.60	–	–	–	–	–	–

Table 11: Runtime (in seconds) in the roadmap (dimacs) domain with brute-force search algorithms

Runtime (seconds) - USA Roadmap dimacs								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	**17.15**	**17.62**	**17.83**	**17.66**	**18.00**	**18.35**	**20.60**	**23.29**
K_0	34.07	34.02	34.32	34.22	34.63	35.08	38.09	42.70

Table 12: Runtime (in seconds) in the roadmap (dimacs) domain with brute-force search algorithms

Runtime (seconds) - W Roadmap dimacs								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	**4.33**	**4.55**	**4.56**	**4.58**	**4.78**	**5.03**	**6.43**	**8.22**
K_0	8.46	8.53	8.72	8.61	8.76	9.00	10.30	12.57

Table 13: Runtime (in seconds) in the roadmap (dimacs) domain with brute-force search algorithms

Runtime (seconds) - BAY Roadmap dimacs								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA^*$	0.37	**0.37**	**0.39**	**0.40**	**0.52**	**0.65**	**1.63**	**2.79**
mA^*	**0.33**	2.37	–	–	–	–	–	–

Table 14: Runtime (in seconds) in the roadmap (dimacs) domain with heuristic search algorithms

Runtime (seconds) - CAL Roadmap dimacs								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA^*$	1.83	**1.87**	**1.90**	**1.93**	**2.12**	**2.36**	**4.15**	6.31
K^*	3.07	3.12	3.15	3.16	3.27	3.37	4.40	**5.80**
mA^*	**1.66**	11.44	–	–	–	–	–	–

Table 15: Runtime (in seconds) in the roadmap (dimacs) domain with heuristic search algorithms

Runtime (seconds) - COL Roadmap dimacs								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA^*$	0.53	**0.53**	**0.55**	**0.57**	**0.69**	**0.84**	1.93	3.24
K^*	0.83	0.84	0.85	0.86	0.92	0.99	**1.65**	**2.50**
mA^*	**0.48**	3.01	–	–	–	–	–	–

Table 16: Runtime (in seconds) in the roadmap (dimacs) domain with heuristic search algorithms

Runtime (seconds) - CTR Roadmap dimacs								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA^*$	**10.84**	**11.06**	**11.19**	**11.38**	**11.61**	**11.80**	**14.06**	**17.03**
K^*	19.70	19.73	20.05	20.00	20.34	20.34	22.54	25.38

Table 17: Runtime (in seconds) in the roadmap (dimacs) domain with heuristic search algorithms

Runtime (seconds) - E Roadmap dimacs								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA^*$	3.31	**3.41**	**3.46**	**3.51**	**3.78**	**4.07**	**6.52**	**9.49**
K^*	5.77	5.83	5.89	5.91	6.09	6.28	8.03	10.52
mA^*	**2.94**	21.63	–	–	–	–	–	–

Table 18: Runtime (in seconds) in the roadmap (dimacs) domain with heuristic search algorithms

Runtime (seconds) - FLA Roadmap dimacs								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
BELA*	1.20	**1.24**	**1.26**	**1.28**	**1.47**	**1.69**	3.41	5.49
K*	2.05	2.09	2.11	2.11	2.23	2.32	**3.30**	**4.64**
mA*	**1.07**	7.37	–	–	–	–	–	–

Table 19: Runtime (in seconds) in the roadmap (dimacs) domain with heuristic search algorithms

Runtime (seconds) - LKS Roadmap dimacs								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
BELA*	2.62	**2.70**	**2.75**	**2.79**	**3.09**	**3.45**	**6.08**	**9.17**
mA*	**2.30**	15.87	–	–	–	–	–	–

Table 20: Runtime (in seconds) in the roadmap (dimacs) domain with heuristic search algorithms

Runtime (seconds) - NE Roadmap dimacs								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
BELA*	1.53	**1.58**	**1.61**	**1.63**	**1.82**	**2.05**	**3.81**	5.91
K*	2.66	2.71	2.73	2.74	2.85	2.95	4.01	**5.44**
mA*	**1.40**	10.47	–	–	–	–	–	–

Table 21: Runtime (in seconds) in the roadmap (dimacs) domain with heuristic search algorithms

Runtime (seconds) - NW Roadmap dimacs								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
BELA*	1.11	**1.14**	**1.16**	**1.18**	**1.36**	**1.60**	3.24	5.24
K*	1.84	1.87	1.89	1.90	1.99	2.09	**2.95**	**4.13**
mA*	**1.00**	6.60	–	–	–	–	–	–

Table 22: Runtime (in seconds) in the roadmap (dimacs) domain with heuristic search algorithms

Runtime (seconds) - NY Roadmap dimacs								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
BELA*	0.39	**0.39**	**0.41**	**0.42**	**0.54**	**0.67**	1.65	2.81
K*	0.65	0.66	0.67	0.67	0.73	0.79	**1.28**	**1.92**
mA*	**0.34**	2.72	–	–	–	–	–	–

Table 23: Runtime (in seconds) in the roadmap (dimacs) domain with heuristic search algorithms

Runtime (seconds) - USA Roadmap dimacs								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
BELA*	**19.17**	**19.52**	**19.63**	**19.84**	**20.25**	**20.73**	**24.73**	**29.46**
K*	35.80	35.91	35.99	36.09	36.32	36.61	39.94	44.42

Table 24: Runtime (in seconds) in the roadmap (dimacs) domain with heuristic search algorithms

Runtime (seconds) - W Roadmap dimacs								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
BELA*	**4.86**	**5.01**	**5.05**	**5.04**	**5.35**	**5.72**	**8.21**	**11.16**
K*	8.78	8.91	8.97	9.00	9.18	9.36	10.76	12.85

Table 25: Runtime (in seconds) in the roadmap (dimacs) domain with heuristic search algorithms

Runtime (seconds) - BAY Roadmap dimacs								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	**0.33**	**0.33**	**0.34**	**0.35**	**0.43**	**0.51**	1.15	1.91
K_0	0.56	0.56	0.57	0.58	0.63	0.68	**1.14**	**1.74**
$BELA^*$	0.37	0.37	0.39	0.40	0.52	0.65	1.63	2.79

Table 26: Runtime (in seconds) in the roadmap (dimacs) domain with mixed search algorithms

Runtime (seconds) - CAL Roadmap dimacs								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	**1.60**	**1.62**	**1.64**	**1.65**	**1.78**	**1.93**	**3.05**	**4.43**
K_0	2.85	2.89	2.93	2.94	3.04	3.16	4.17	5.58
$BELA^*$	1.83	1.87	1.90	1.93	2.12	2.36	4.15	6.31
K^*	3.07	3.12	3.15	3.16	3.27	3.37	4.40	5.80

Table 27: Runtime (in seconds) in the roadmap (dimacs) domain with mixed search algorithms

Runtime (seconds) - COL Roadmap dimacs								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	**0.47**	**0.48**	**0.49**	**0.50**	**0.58**	**0.68**	**1.37**	**2.21**
K_0	0.76	0.77	0.78	0.79	0.85	0.92	1.58	2.44
$BELA^*$	0.53	0.53	0.55	0.57	0.69	0.84	1.93	3.24
K^*	0.83	0.84	0.85	0.86	0.92	0.99	1.65	2.50

Table 28: Runtime (in seconds) in the roadmap (dimacs) domain with mixed search algorithms

Runtime (seconds) - CTR Roadmap dimacs								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	**9.85**	**10.04**	**10.17**	**10.17**	**10.48**	**10.52**	**12.01**	**13.85**
K_0	18.97	19.09	19.16	19.31	19.50	19.54	21.73	24.63
$BELA^*$	10.84	11.06	11.19	11.38	11.61	11.80	14.06	17.03
K^*	19.70	19.73	20.05	20.00	20.34	20.34	22.54	25.38

Table 29: Runtime (in seconds) in the roadmap (dimacs) domain with mixed search algorithms

Runtime (seconds) - E Roadmap dimacs								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	**2.81**	**2.85**	**2.90**	**2.92**	**3.08**	**3.27**	**4.72**	**6.51**
K_0	5.39	5.43	5.47	5.51	5.68	5.88	7.66	10.15
$BELA^*$	3.31	3.41	3.46	3.51	3.78	4.07	6.52	9.49
K^*	5.77	5.83	5.89	5.91	6.09	6.28	8.03	10.52

Table 30: Runtime (in seconds) in the roadmap (dimacs) domain with mixed search algorithms

Runtime (seconds) - FLA Roadmap dimacs								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	**1.05**	**1.07**	**1.08**	**1.10**	**1.22**	**1.36**	**2.45**	**3.75**
K_0	1.87	1.90	1.92	1.93	2.03	2.13	3.13	4.43
$BELA^*$	1.20	1.24	1.26	1.28	1.47	1.69	3.41	5.49
K^*	2.05	2.09	2.11	2.11	2.23	2.32	3.30	4.64

Table 31: Runtime (in seconds) in the roadmap (dimacs) domain with mixed search algorithms

Runtime (seconds) - LKS Roadmap dimacs								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	**2.27**	**2.33**	**2.38**	**2.40**	**2.59**	**2.80**	**4.41**	**6.33**
K_0	4.14	4.20	4.27	4.28	4.49	4.71	6.65	9.29
$BELA^*$	2.62	2.70	2.75	2.79	3.09	3.45	6.08	9.17

Table 32: Runtime (in seconds) in the roadmap (dimacs) domain with mixed search algorithms

Runtime (seconds) - NE Roadmap dimacs								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	**1.34**	**1.37**	**1.40**	**1.41**	**1.53**	**1.68**	**2.79**	**4.09**
K_0	2.44	2.48	2.50	2.52	2.62	2.73	3.79	5.21
$BELA^*$	1.53	1.58	1.61	1.63	1.82	2.05	3.81	5.91
K^*	2.66	2.71	2.73	2.74	2.85	2.95	4.01	5.44

Table 33: Runtime (in seconds) in the roadmap (dimacs) domain with mixed search algorithms

Runtime (seconds) - NW Roadmap dimacs								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	**1.02**	**1.04**	**1.06**	**1.08**	**1.19**	**1.33**	**2.37**	**3.62**
K_0	1.74	1.77	1.79	1.79	1.89	1.99	2.85	4.03
$BELA^*$	1.11	1.14	1.16	1.18	1.36	1.60	3.24	5.24
K^*	1.84	1.87	1.89	1.90	1.99	2.09	2.95	4.13

Table 34: Runtime (in seconds) in the roadmap (dimacs) domain with mixed search algorithms

Runtime (seconds) - NY Roadmap dimacs								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	**0.32**	**0.34**	**0.34**	**0.35**	**0.42**	**0.50**	**1.11**	**1.82**
K_0	0.59	0.59	0.61	0.62	0.66	0.72	1.20	1.84
$BELA^*$	0.39	0.39	0.41	0.42	0.54	0.67	1.65	2.81
K^*	0.65	0.66	0.67	0.67	0.73	0.79	1.28	1.92

Table 35: Runtime (in seconds) in the roadmap (dimacs) domain with mixed search algorithms

Runtime (seconds) - USA Roadmap dimacs								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	**17.15**	**17.62**	**17.83**	**17.66**	**18.00**	**18.35**	**20.60**	**23.29**
K_0	34.07	34.02	34.32	34.22	34.63	35.08	38.09	42.70
$BELA^*$	19.17	19.52	19.63	19.84	20.25	20.73	24.73	29.46
K^*	35.80	35.91	35.99	36.09	36.32	36.61	39.94	44.42

Table 36: Runtime (in seconds) in the roadmap (dimacs) domain with mixed search algorithms

Runtime (seconds) - W Roadmap dimacs								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	**4.33**	**4.55**	**4.56**	**4.58**	**4.78**	**5.03**	**6.43**	**8.22**
K_0	8.46	8.53	8.72	8.61	8.76	9.00	10.30	12.57
$BELA^*$	4.86	5.01	5.05	5.04	5.35	5.72	8.21	11.16
K^*	8.78	8.91	8.97	9.00	9.18	9.36	10.76	12.85

Table 37: Runtime (in seconds) in the roadmap (dimacs) domain with mixed search algorithms

Runtime (seconds) - BAY Roadmap unit								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	0.23	**0.23**	**0.24**	**0.24**	**0.25**	**0.25**	**0.31**	**0.38**
K_0	0.50	0.51	0.51	0.51	0.54	0.57	0.86	1.22
mDijkstra	**0.22**	1.64	–	–	–	–	–	–

Table 38: Runtime (in seconds) in the roadmap (unit) domain with brute-force search algorithms

Runtime (seconds) - CAL Roadmap unit								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	1.15	**1.16**	**1.17**	**1.17**	**1.20**	**1.21**	**1.32**	**1.46**
K_0	2.49	2.50	2.52	2.52	2.59	2.66	3.40	4.34
mDijkstra	**1.04**	7.92	–	–	–	–	–	–

Table 39: Runtime (in seconds) in the roadmap (unit) domain with brute-force search algorithms

Runtime (seconds) - COL Roadmap unit								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	0.29	**0.29**	**0.29**	**0.29**	**0.30**	**0.31**	**0.41**	**0.53**
K_0	0.59	0.60	0.61	0.61	0.65	0.70	1.15	1.71
mDijkstra	**0.26**	1.90	–	–	–	–	–	–

Table 40: Runtime (in seconds) in the roadmap (unit) domain with brute-force search algorithms

Runtime (seconds) - E Roadmap unit								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	2.15	**2.17**	**2.22**	**2.22**	**2.24**	**2.26**	**2.43**	**2.65**
K_0	4.79	4.81	4.84	4.84	4.96	5.07	6.28	7.87
mDijkstra	**2.05**	15.85	–	–	–	–	–	–

Table 41: Runtime (in seconds) in the roadmap (unit) domain with brute-force search algorithms

Runtime (seconds) - FLA Roadmap unit								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	0.74	**0.77**	**0.77**	**0.77**	**0.78**	**0.79**	**0.88**	**1.00**
K_0	1.63	1.64	1.66	1.66	1.71	1.77	2.31	3.00
mDijkstra	**0.67**	5.34	–	–	–	–	–	–

Table 42: Runtime (in seconds) in the roadmap (unit) domain with brute-force search algorithms

Runtime (seconds) - LKS Roadmap unit								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	1.58	**1.63**	**1.63**	**1.64**	**1.65**	**1.67**	**1.81**	**2.00**
K_0	3.56	3.58	3.60	3.61	3.72	3.84	5.05	6.78
mDijkstra	**1.41**	10.73	–	–	–	–	–	–

Table 43: Runtime (in seconds) in the roadmap (unit) domain with brute-force search algorithms

Runtime (seconds) - NE Roadmap unit								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	0.97	**0.99**	**1.00**	**1.00**	**1.01**	**1.02**	**1.13**	**1.27**
K_0	2.18	2.19	2.21	2.21	2.29	2.36	3.09	4.02
mDijkstra	**0.86**	6.76	–	–	–	–	–	–

Table 44: Runtime (in seconds) in the roadmap (unit) domain with brute-force search algorithms

Runtime (seconds) - NW Roadmap unit								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	0.68	**0.69**	**0.69**	**0.70**	**0.71**	**0.72**	**0.83**	**0.97**
K_0	1.48	1.49	1.50	1.51	1.56	1.64	2.29	3.13
mDijkstra	**0.62**	4.52	–	–	–	–	–	–

Table 45: Runtime (in seconds) in the roadmap (unit) domain with brute-force search algorithms

Runtime (seconds) - NY Roadmap unit								
Algorithm	k=1	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	0.23	**0.23**	**0.24**	**0.23**	**0.24**	**0.25**	**0.30**	**0.36**
K_0	0.50	0.50	0.51	0.51	0.54	0.57	0.87	1.25
mDijkstra	**0.21**	1.60	–	–	–	–	–	–

Table 46: Runtime (in seconds) in the roadmap (unit) domain with brute-force search algorithms

Runtime (seconds) - Maps 10 unit												
Algorithm	k=1	k=2	k=3	k=4	k=5	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	**0.22**	**0.22**	**0.22**	**0.22**	**0.22**	**0.22**	**0.23**	**0.22**	**0.23**	**0.23**	**0.28**	**0.34**
K_0	0.72	0.71	0.72	0.72	0.72	0.72	0.73	0.74	0.86	1.00	2.14	4.25
mDijkstra	0.26	0.48	0.68	0.88	1.09	2.15	14.53	42.94	–	–	–	–

Table 47: Runtime (in seconds) in the maps (unit) domain with brute-force search algorithms

Runtime (seconds) - Maps 15 unit												
Algorithm	k=1	k=2	k=3	k=4	k=5	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	0.28	**0.28**	**0.28**	**0.28**	**0.28**	**0.28**	**0.28**	**0.28**	**0.29**	**0.29**	**0.37**	**0.46**
K_0	0.67	0.66	0.66	0.67	0.67	0.67	0.68	0.69	0.82	0.94	2.07	4.14
mDijkstra	**0.25**	0.44	0.63	0.81	1.01	1.96	13.17	39.12	–	–	–	–

Table 48: Runtime (in seconds) in the maps (unit) domain with brute-force search algorithms

Runtime (seconds) - Maps 20 unit												
Algorithm	k=1	k=2	k=3	k=4	k=5	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	0.26	**0.26**	**0.26**	**0.26**	**0.26**	**0.26**	**0.26**	**0.26**	**0.27**	**0.27**	**0.35**	**0.42**
K_0	0.62	0.62	0.62	0.62	0.62	0.63	0.63	0.65	0.79	0.90	2.02	4.09
mDijkstra	**0.23**	0.40	0.57	0.73	0.90	1.75	11.43	34.39	–	–	–	–

Table 49: Runtime (in seconds) in the maps (unit) domain with brute-force search algorithms

Runtime (seconds) - Maps 25 unit												
Algorithm	k=1	k=2	k=3	k=4	k=5	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	**0.18**	**0.18**	**0.18**	**0.18**	**0.18**	**0.18**	**0.18**	**0.18**	**0.18**	**0.19**	**0.24**	**0.29**
K_0	0.58	0.58	0.58	0.58	0.58	0.59	0.60	0.61	0.73	0.84	1.96	4.02
mDijkstra	0.22	0.38	0.53	0.68	0.83	1.60	10.18	31.07	–	–	–	–

Table 50: Runtime (in seconds) in the maps (unit) domain with brute-force search algorithms

Runtime (seconds) - Maps 30 unit												
Algorithm	k=1	k=2	k=3	k=4	k=5	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	0.22	**0.22**	**0.22**	**0.22**	**0.22**	**0.22**	**0.22**	**0.22**	**0.23**	**0.23**	**0.31**	**0.39**
K_0	0.46	0.46	0.46	0.46	0.47	0.47	0.48	0.49	0.58	0.68	1.80	3.85
mDijkstra	**0.20**	0.33	0.46	0.59	0.72	1.38	8.71	25.91	–	–	–	–

Table 51: Runtime (in seconds) in the maps (unit) domain with brute-force search algorithms

Runtime (seconds) - Maps 35 unit												
Algorithm	k=1	k=2	k=3	k=4	k=5	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	0.18	**0.18**	**0.18**	**0.18**	**0.18**	**0.18**	**0.18**	**0.18**	**0.19**	**0.19**	**0.27**	**0.34**
K_0	0.40	0.40	0.40	0.40	0.40	0.40	0.41	0.42	0.51	0.63	1.82	3.96
mDijkstra	**0.16**	0.28	0.39	0.50	0.61	1.18	7.31	21.67	–	–	–	–

Table 52: Runtime (in seconds) in the maps (unit) domain with brute-force search algorithms

Runtime (seconds) - Maps 10 unit												
Algorithm	k=1	k=2	k=3	k=4	k=5	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA^*$	0.01	**0.01**	**0.01**	**0.01**	**0.01**	**0.01**	**0.01**	**0.01**	**0.01**	**0.02**	**0.04**	**0.08**
K^*	0.06	1.60	1.79	1.93	1.94	1.94	1.96	1.96	2.07	2.16	3.03	4.16
mA^*	**0.01**	0.01	0.02	0.03	0.03	0.06	0.32	0.90	8.06	–	–	–

Table 53: Runtime (in seconds) in the maps (unit) domain with heuristic search algorithms

Runtime (seconds) - Maps 15 unit												
Algorithm	k=1	k=2	k=3	k=4	k=5	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA^*$	0.02	0.02	**0.02**	**0.02**	**0.02**	**0.02**	**0.02**	**0.02**	**0.03**	**0.03**	**0.06**	**0.11**
K^*	0.06	1.25	1.52	1.65	1.65	1.65	1.67	1.68	1.79	1.88	2.75	3.90
mA^*	**0.01**	**0.02**	0.03	0.03	0.04	0.08	0.43	1.18	10.41	–	–	–

Table 54: Runtime (in seconds) in the maps (unit) domain with heuristic search algorithms

Runtime (seconds) - Maps 20 unit												
Algorithm	k=1	k=2	k=3	k=4	k=5	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA^*$	0.04	0.04	**0.04**	**0.04**	**0.04**	**0.04**	**0.04**	**0.04**	**0.04**	**0.04**	**0.08**	**0.14**
K^*	0.09	1.02	1.17	1.31	1.31	1.31	1.33	1.34	1.44	1.53	2.41	3.63
mA^*	**0.02**	**0.04**	0.05	0.07	0.08	0.16	0.87	2.42	21.51	–	–	–

Table 55: Runtime (in seconds) in the maps (unit) domain with heuristic search algorithms

Runtime (seconds) - Maps 25 unit												
Algorithm	k=1	k=2	k=3	k=4	k=5	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA^*$	0.04	0.04	**0.04**	**0.04**	**0.04**	**0.04**	**0.04**	**0.04**	**0.05**	**0.05**	**0.09**	**0.16**
K^*	0.09	0.82	1.03	1.08	1.08	1.08	1.10	1.11	1.21	1.30	2.24	3.60
mA^*	**0.02**	**0.04**	0.05	0.07	0.09	0.17	0.91	2.55	22.95	–	–	–

Table 56: Runtime (in seconds) in the maps (unit) domain with heuristic search algorithms

Runtime (seconds) - Maps 30 unit												
Algorithm	k=1	k=2	k=3	k=4	k=5	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA^*$	0.02	**0.02**	**0.02**	**0.02**	**0.02**	**0.02**	**0.03**	**0.02**	**0.03**	**0.03**	**0.07**	**0.12**
K^*	0.08	0.31	0.34	0.35	0.35	0.35	0.36	0.37	0.46	0.55	1.59	3.41
mA^*	**0.02**	0.04	0.05	0.07	0.09	0.17	0.91	2.51	23.71	–	–	–

Table 57: Runtime (in seconds) in the maps (unit) domain with heuristic search algorithms

Runtime (seconds) - Maps 35 unit												
Algorithm	k=1	k=2	k=3	k=4	k=5	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA^*$	0.06	**0.06**	**0.06**	**0.06**	**0.06**	**0.06**	**0.06**	**0.06**	**0.07**	**0.07**	**0.12**	**0.19**
K^*	0.14	0.15	0.15	0.15	0.15	0.15	0.16	0.17	0.25	0.35	1.44	3.48
mA^*	**0.03**	0.06	0.09	0.12	0.15	0.29	1.63	4.65	44.51	–	–	–

Table 58: Runtime (in seconds) in the maps (unit) domain with heuristic search algorithms

Runtime (seconds) - Maps 10 octile												
Algorithm	k=1	k=2	k=3	k=4	k=5	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	**0.46**	**0.46**	**0.46**	**0.47**	**0.47**	**0.46**	**0.47**	**0.47**	**0.47**	**0.47**	**0.51**	**0.55**
K_0	1.50	1.49	1.50	1.50	1.50	1.50	1.51	1.52	1.57	1.62	2.33	3.48
mDijkstra	0.52	0.96	1.37	1.80	2.33	5.06	31.55	94.43	–	–	–	–

Table 59: Runtime (in seconds) in the maps (octile) domain with brute-force search algorithms

Runtime (seconds) - Maps 15 octile												
Algorithm	k=1	k=2	k=3	k=4	k=5	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	0.59	**0.59**	**0.59**	**0.59**	**0.59**	**0.59**	**0.59**	**0.59**	**0.59**	**0.60**	**0.66**	**0.72**
K_0	1.25	1.25	1.25	1.25	1.26	1.25	1.26	1.26	1.32	1.37	2.10	3.34
mDijkstra	**0.47**	0.85	1.21	1.58	2.01	4.32	27.50	83.26	–	–	–	–

Table 60: Runtime (in seconds) in the maps (octile) domain with brute-force search algorithms

Runtime (seconds) - Maps 20 octile												
Algorithm	k=1	k=2	k=3	k=4	k=5	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	0.54	**0.54**	**0.54**	**0.54**	**0.54**	**0.54**	**0.54**	**0.54**	**0.54**	**0.55**	**0.61**	**0.68**
K_0	1.14	1.14	1.14	1.14	1.14	1.14	1.15	1.16	1.21	1.27	1.99	3.31
mDijkstra	**0.42**	0.76	1.08	1.40	1.75	3.71	24.15	73.40	–	–	–	–

Table 61: Runtime (in seconds) in the maps (octile) domain with brute-force search algorithms

Runtime (seconds) - Maps 25 octile												
Algorithm	k=1	k=2	k=3	k=4	k=5	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	0.48	**0.48**	**0.48**	**0.48**	**0.48**	**0.48**	**0.48**	**0.48**	**0.49**	**0.49**	**0.53**	**0.58**
K_0	1.05	1.05	1.05	1.05	1.05	1.05	1.06	1.06	1.12	1.17	1.92	3.22
mDijkstra	**0.38**	0.68	0.96	1.24	1.55	3.20	20.96	63.91	–	–	–	–

Table 62: Runtime (in seconds) in the maps (octile) domain with brute-force search algorithms

Runtime (seconds) - Maps 30 octile												
Algorithm	k=1	k=2	k=3	k=4	k=5	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	0.43	**0.43**	**0.43**	**0.43**	**0.43**	**0.43**	**0.43**	**0.43**	**0.44**	**0.43**	**0.48**	**0.54**
K_0	0.95	0.94	0.95	0.95	0.95	0.95	0.96	0.96	1.01	1.07	1.83	3.20
mDijkstra	**0.34**	0.60	0.85	1.09	1.35	2.72	18.01	55.21	–	–	–	–

Table 63: Runtime (in seconds) in the maps (octile) domain with brute-force search algorithms

Runtime (seconds) - Maps 35 octile												
Algorithm	k=1	k=2	k=3	k=4	k=5	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	**0.27**	**0.27**	**0.27**	**0.27**	**0.27**	**0.27**	**0.27**	**0.27**	**0.27**	**0.27**	**0.30**	**0.32**
K_0	0.75	0.75	0.75	0.75	0.76	0.76	0.76	0.77	0.83	0.89	1.68	3.12
mDijkstra	0.29	0.52	0.73	0.93	1.16	2.30	15.32	47.21	–	–	–	–

Table 64: Runtime (in seconds) in the maps (octile) domain with brute-force search algorithms

Runtime (seconds) - Maps 10 octile												
Algorithm	k=1	k=2	k=3	k=4	k=5	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA^*$	**0.01**	**0.01**	**0.01**	**0.01**	**0.01**	**0.01**	**0.01**	**0.01**	**0.01**	**0.02**	**0.04**	**0.07**
K^*	0.05	0.22	0.23	0.23	0.24	0.24	0.26	0.26	0.31	0.36	0.97	2.09
mA^*	0.02	0.03	0.04	0.06	0.07	0.14	0.77	2.16	–	–	–	–

Table 65: Runtime (in seconds) in the maps (octile) domain with heuristic search algorithms

Runtime (seconds) - Maps 15 octile												
Algorithm	k=1	k=2	k=3	k=4	k=5	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
BELA*	0.04	**0.04**	**0.04**	**0.04**	**0.04**	**0.04**	**0.04**	**0.04**	**0.04**	**0.05**	**0.08**	**0.13**
K*	0.07	0.10	0.11	0.11	0.11	0.12	0.13	0.13	0.18	0.23	0.89	2.14
mA*	**0.03**	0.06	0.08	0.11	0.14	0.28	1.67	4.67	–	–	–	–

Table 66: Runtime (in seconds) in the maps (octile) domain with heuristic search algorithms

Runtime (seconds) - Maps 20 octile												
Algorithm	k=1	k=2	k=3	k=4	k=5	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
BELA*	0.05	**0.05**	**0.05**	**0.05**	**0.05**	**0.05**	**0.05**	**0.05**	**0.05**	**0.05**	**0.10**	**0.13**
K*	0.08	0.10	0.10	0.10	0.11	0.11	0.12	0.13	0.17	0.22	0.92	2.29
mA*	**0.04**	0.08	0.11	0.15	0.18	0.37	2.18	5.98	–	–	–	–

Table 67: Runtime (in seconds) in the maps (octile) domain with heuristic search algorithms

Runtime (seconds) - Maps 25 octile												
Algorithm	k=1	k=2	k=3	k=4	k=5	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
BELA*	0.05	**0.05**	**0.05**	**0.05**	**0.06**	**0.05**	**0.06**	**0.06**	**0.06**	**0.06**	**0.10**	**0.14**
K*	0.10	0.11	0.11	0.12	0.12	0.12	0.13	0.13	0.18	0.24	0.94	2.27
mA*	**0.05**	0.09	0.13	0.18	0.22	0.44	2.63	7.34	–	–	–	–

Table 68: Runtime (in seconds) in the maps (octile) domain with heuristic search algorithms

Runtime (seconds) - Maps 30 octile												
Algorithm	k=1	k=2	k=3	k=4	k=5	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
BELA*	**0.04**	**0.04**	**0.04**	**0.04**	**0.04**	**0.04**	**0.04**	**0.04**	**0.04**	**0.05**	**0.07**	**0.10**
K*	0.12	0.13	0.13	0.13	0.13	0.13	0.14	0.15	0.19	0.25	0.98	2.36
mA*	0.05	0.10	0.15	0.20	0.25	0.50	3.00	8.38	–	–	–	–

Table 69: Runtime (in seconds) in the maps (octile) domain with heuristic search algorithms

Runtime (seconds) - Maps 35 octile												
Algorithm	k=1	k=2	k=3	k=4	k=5	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
BELA*	0.07	**0.07**	**0.07**	**0.07**	**0.07**	**0.07**	**0.07**	**0.07**	**0.07**	**0.07**	**0.12**	**0.17**
K*	0.12	0.13	0.13	0.14	0.14	0.14	0.15	0.16	0.20	0.27	1.02	2.42
mA*	**0.06**	0.11	0.16	0.21	0.26	0.54	3.17	9.09	–	–	–	–

Table 70: Runtime (in seconds) in the maps (octile) domain with heuristic search algorithms

Runtime (seconds) - 10-Pancake heavy-cost						
Algorithm	k=1	k=2	k=3	k=4	k=5	k=10
$BELA_0$	29.05	**29.92**	**30.69**	**31.04**	**31.39**	**32.31**
K_0	79.06	87.47	91.24	92.22	93.03	–
mDijkstra	**25.88**	52.56	–	–	–	–

Table 71: Runtime (in seconds) in the n-pancake (unit) domain with brute-force search algorithms

Runtime (seconds) - 20-Pancake unit													
Algorithm	k=1	k=2	k=3	k=4	k=5	k=10	k=20	k=40	k=50	k=100	k=500	k=900	k=1000
BELA*	0.00	**0.00**	**0.01**	**0.01**	**0.01**	**0.01**	**0.02**	**0.04**	**0.04**	**0.07**	**0.24**	**0.38**	**0.43**
K*	0.02	1.44	1.77	2.65	2.91	4.83	–	–	–	–	–	–	–
mA*	**0.00**	0.00	0.01	0.01	0.01	0.02	0.03	0.05	0.06	0.12	0.54	0.96	1.05

Table 72: Runtime (in seconds) in the n-pancake (unit) domain with heuristic search algorithms

Runtime (seconds) - 30-Pancake unit												
Algorithm	k=1	k=2	k=3	k=4	k=5	k=10	k=20	k=40	k=50	k=100	k=500	k=900
BELA*	0.05	**0.08**	**0.11**	**0.13**	**0.16**	**0.28**	**0.43**	**0.66**	**0.74**	**1.24**	**3.47**	**4.49**
K*	0.41	–	–	–	–	–	–	–	–	–	–	–
mA*	**0.05**	0.09	0.12	0.14	0.18	0.30	0.48	0.85	0.99	1.70	7.36	–

Table 73: Runtime (in seconds) in the n-pancake (unit) domain with heuristic search algorithms

Runtime (seconds) - 40-Pancake unit								
Algorithm	k=1	k=2	k=3	k=4	k=5	k=10	k=20	k=40
BELA*	0.61	**0.92**	**1.19**	**1.48**	**1.72**	**2.89**	**4.50**	**7.16**
K*	6.12	–	–	–	–	–	–	–
mA*	**0.60**	0.92	1.27	1.64	1.90	3.19	5.30	–

Table 74: Runtime (in seconds) in the n-pancake (unit) domain with heuristic search algorithms

A.3.2 Heavy-cost variant

The results in the heavy-cost variant are shown in Tables 75–76. The first table shows the results of the brute-force variants. The last table shows the results of the heuristic search algorithms. Due to its difficulty, only the 10-Pancake was used.

A.4 N-Puzzle

The runtime of all algorithms being tested is shown in Tables 77–81, both in the 8- and 15-Puzzle.

A.4.1 Unit variant

Table 77 shows the runtime of the brute-force search algorithms in the 8-Puzzle. Tables 78–79 show the runtime of the heuristic search algorithms in the 8- and the 15-Puzzle respectively, in the unit variant.

A.4.2 Heavy-cost variant

The heavy-cost variant has being tried only with the heuristic variants. Tables 80–81 show the runtime of all algorithms being tested.

Runtime (seconds) - 10-Pancake heavy-cost						
Algorithm	k=1	k=2	k=3	k=4	k=5	k=10
BELA_0	29.05	**29.92**	**30.69**	**31.04**	**31.39**	**32.31**
K_0	79.06	87.47	91.24	92.22	93.03	–
mDijkstra	**25.88**	52.56	–	–	–	–

Table 75: Runtime (in seconds) in the n-pancake (heavy-cost) domain with brute-force search algorithms

Runtime (seconds) - 10-Pancake heavy-cost									
Algorithm	k=1	k=2	k=3	k=4	k=5	k=10	k=40	k=50	k=100
BELA*	0.39	**0.41**	**0.42**	**0.43**	**0.44**	**0.47**	**0.58**	**0.61**	**0.67**
K*	1.99	2.47	3.04	3.32	3.44	4.85	–	–	–
mA*	**0.38**	0.74	1.10	1.41	1.79	3.50	–	–	–

Table 76: Runtime (in seconds) in the n-pancake (heavy-cost) domain with heuristic search algorithms

Runtime (seconds) - 8-Puzzle unit												
Algorithm	k=1	k=2	k=3	k=4	k=5	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
$BELA_0$	0.22	**0.25**	**0.27**	**0.28**	**0.29**	**0.30**	**0.33**	**0.37**	**0.41**	**0.42**	**0.53**	**0.59**
K_0	0.52	0.67	0.71	0.73	0.75	0.77	0.83	0.87	0.92	0.93	1.03	1.11
mDijkstra	**0.20**	0.42	0.64	0.84	1.03	1.83	–	–	–	–	–	–

Table 77: Runtime (in seconds) in the n-puzzle (unit) domain with brute-force search algorithms

Runtime (seconds) - 8-Puzzle heavy-cost												
Algorithm	k=1	k=2	k=3	k=4	k=5	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
BELA*	0.00	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.01**	**0.01**	**0.02**	**0.08**	**0.15**
K*	0.01	0.01	0.01	0.02	0.02	0.02	0.03	0.03	0.04	0.05	0.13	0.21
mA*	**0.00**	0.00	0.01	0.01	0.01	0.01	0.05	0.12	0.86	2.49	–	–

Table 78: Runtime (in seconds) in the n-puzzle (unit) domain with heuristic search algorithms

Runtime (seconds) - 15-Puzzle heavy-cost						
Algorithm	k=1	k=2	k=3	k=4	k=5	k=10
BELA*	5.26	**5.43**	**5.75**	**5.74**	**5.81**	**6.17**
K*	18.58	–	–	–	–	–
mA*	**4.98**	8.95	13.59	–	–	–

Table 79: Runtime (in seconds) in the n-puzzle (unit) domain with heuristic search algorithms

Runtime (seconds) - 8-Puzzle heavy-cost												
Algorithm	k=1	k=2	k=3	k=4	k=5	k=10	k=50	k=100	k=500	k=1000	k=5000	k=10000
BELA*	0.00	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.01**	**0.01**	**0.02**	**0.08**	**0.15**
K*	0.01	0.01	0.01	0.02	0.02	0.02	0.03	0.03	0.04	0.05	0.13	0.21
mA*	**0.00**	0.00	0.01	0.01	0.01	0.01	0.05	0.12	0.86	2.49	–	–

Table 80: Runtime (in seconds) in the n-puzzle (heavy-cost) domain with heuristic search algorithms

Runtime (seconds) - 15-Puzzle heavy-cost						
Algorithm	k=1	k=2	k=3	k=4	k=5	k=10
BELA*	5.26	**5.43**	**5.75**	**5.74**	**5.81**	**6.17**
K*	18.58	–	–	–	–	–
mA*	**4.98**	8.95	13.59	–	–	–

Table 81: Runtime (in seconds) in the n-puzzle (heavy-cost) domain with heuristic search algorithms

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